

On the Black-Hole/Qubit Correspondence

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ABSTRACT

The entanglement classification of four qubits is related to the extremal black holes of the 4-dimensional STU model via a time-like reduction to three dimensions. This correspondence is generalised to the entanglement classification of a very special four-way entanglement of eight qubits and the black holes of the maximally supersymmetric $\mathcal{N} = 8$ and exceptional magic $\mathcal{N} = 2$ supergravity theories.

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1 Introduction

The advent of quantum theory heralded a new era of understanding - we inhabit a fundamentally probabilistic world founded upon the principle of quantum superposition. Since *information* is stored, processed and distributed by physical phenomena, such a radical reassessment of reality ought to carry with it some profound implications for our theories of information and computation. A concerted effort to understand these implications swiftly developed into the fascinating and rapidly expanding field of *quantum information theory* (QIT) [1]. One of the principal goals of QIT is to characterise the behaviour and computational potential of information processing systems which utilise the fundamental properties of quantum mechanics. There is an expectation that quantum theory may be exploited to perform computational tasks beyond the capability of any, even idealistic, purely classical device. This possibility enjoys a certain poetry: just as the conventional microchip meets its fundamental limit, fixed by the onset of quantum noise at the atomic scale, the very same quantum phenomena open the door to new, superior, modes of computation. A key component of today’s quantum information toolkit is the quintessentially quantum phenomenon of *entanglement*. The quantum states of two or more entangled objects must be described with reference to each other, even though the individual objects may be spatially separated. This leads to classically inexplicable, but experimentally observable, quantum correlations between the spatially separated systems - “spooky” action at a distance as Einstein described it. Quantum entanglement is vital to the emerging technologies of quantum computing, communication and cryptography. One of the longest standing open problems in QIT is a complete qualitative and quantitative characterisation of multipartite entanglement.

In quite separate developments Black Holes (BHs) have commanded an equally privileged position in the various attempts to unify the fundamental interactions including quantum gravity. While general relativity refuses to succumb to quantum rule, BHs raise quandaries that strike at the very heart of quantum theory. Without a proper theory of quantum gravity, such paradoxes will continue to haunt us. M-theory, which grew out of pioneering work on supergravity and superstring theory, is a promising approach to quantum gravity. Living in eleven spacetime dimensions, it encompasses and connects the five consistent 10-dimensional superstring theories, as well as 11-dimensional supergravity and, as such, has the potential to unify the fundamental forces into a single consistent framework. However, M-theory is fundamentally non-perturbative and consequently remains largely mysterious, offering up only remote corners of its full structure. The physics of BHs has occupied centre stage, providing unique insights into the non-perturbative structure of M-theory. Whatever final formulation M-theory eventually takes, understanding its BH solutions will play an essential role in its evolution.

For the most part these important endeavors in quantum information and gravity have led separate lives. However, the present work centres on a curious and unexpected interplay between these seemingly disparate themes. It constitutes one

corner of the *black-hole/qubit correspondence*: a relationship between the entanglement of qubits, the basic units of quantum information, and the entropy of BHs in M-theory. This story began in 2006 [2] when it was observed that the entropy of the *STU* BH [3–5], which appears in the compactification of M-theory to four dimensions, is given by *Cayley’s hyperdeterminant* [6]. Remarkably, the 3-tangle [7], which measures the entanglement shared by three qubits, is also given by Cayley’s hyperdeterminant [8]. It was soon realised that there is in fact a one-to-one correspondence between the classification of 3-qubit entanglement [9] and the classification of extremal *STU* BHs [10]. Further work [11–23] has led to a more complete dictionary translating a variety of phenomena in one language to those in the other. It seems that we are, as yet, only glimpsing the tip of an iceberg.

Here we develop further a recent application [24] of the black-hole/qubit correspondence to the much more difficult problem of classifying 4-qubit entanglement. The experimental significance of this challenge is re-enforced as 4-qubit entanglement is now achievable in the laboratory [25–27]. The key technical ingredient is the Kostant-Sekiguchi theorem [28, 29], which provides the link between the BHs and qubits. Our main result, summarized in Table 2, is that there are 31 entanglement families which reduce to nine up to permutations of the four qubits. Consulting Table 1 we see that the nine agrees with [30, 31], while the 31 is new. From the BH perspective, we find that the attractor equations, which determine the amount of supersymmetry preserved by a particular BH solution, display a symmetry consistent with permutations of the qubits. For example, the *A*-GHZ state yields a set of attractor equations which are invariant under a triality corresponding to the permutation of *B, C, D* in the GHZ state.

We begin, in section 2, with an elementary introduction to entanglement in QIT, with particular emphasis on *Stochastic Local Operations and Classical Communication* and the status of 4-qubit entanglement classification. In section 3 we briefly review BHs in supergravity and, in particular, the role of time-like dimensional reduction and nilpotent orbits. In section 4 we invoke the Kostant-Sekiguchi theorem, which maps the BH solutions to the 4-qubit entanglement classes and provide a detailed analysis of both the structure of the BH solutions and the entanglement classes. We conclude in Sects. 5 and 6 with the generalisation to $\mathcal{N} = 8$ and $\mathcal{N} = 2$ exceptional supergravities, respectively; while admitting a QIT interpretation as the four-way entanglement of eight qubits, these theories are not amenable to the Kostant-Sekiguchi theorem.

2 Entanglement and SLOCC

In their seminal 1935 work Einstein, Podolsky, and Rosen (EPR) correctly concluded that assuming “local realism” the quantum mechanical wave function cannot provide a complete description of physical reality [32]. Entanglement was identified as the chief culprit. They speculated on the existence of a more fundamental underlying (classical) theory that toed the line of local realism. However, such questions remained a matter of philosophical preference, seemingly inaccessible to experiment. All this changed in 1964 when Bell introduced his now famous inequality [33]. In one fell swoop, entanglement had been elevated from a conceptual puzzle to an experimental observable confronting the very assumptions of local realism. This was Bell’s great insight - to derive from EPR’s criteria something which could be used to check experimentally the phenomenological viability of local realism. Moreover, the Bell inequality opened the door to utilising entanglement in quantum information theoretic processes. For example, it famously forms the basis of a secure cryptographic key distribution protocol [34].

As quantum information theory developed, the role of entanglement became increasingly central. Entanglement may be created, manipulated and consumed in the course of a given quantum computation or protocol. Furthermore, it can in fact exist in physically distinct forms. For example, multipartite states provide so-called Bell inequalities without the inequality [35]. All this motivated a pressing need to properly quantify and classify entanglement. Conventionally, the state of a composite system is said to be entangled if it cannot be written as a tensor product of states of the constituent subsystems. However, this particular measure is perhaps insufficient to really capture the various subtleties of entanglement. For example, there are two totally non-separable 3-qubit states that have physically distinct entanglement properties [9]. Is there a more illuminating notion of entanglement? Let us take our cues from experiment. We do not actually observe the tensor product structure, even though it underpins our theoretical understanding. What we do observe are correlations between spatially separated systems that admit no classical explanation. This motivates the more general and quantum information theoretic notion of entanglement as correlations between constituent pieces of a composite system that are of a quantum origin [36–38]. The question now is, how does one differentiate between classical correlations and those correlations which may be attributed to genuine quantum phenomena? Classical correlations are *defined* as those which may be generated by *Local Operations and Classical Communication* (LOCC) [36–38]. Any classical correlation may be experimentally established using LOCC. Conversely, all correlations unobtainable via LOCC are regarded as *bona fide* quantum entanglement.

The LOCC paradigm is quite intuitive. Heuristically, given a composite quantum system with its components spread among different laboratories around the world, one allows each experimenter to perform any quantum operation or measurement on their component locally in their laboratory. These local operations cannot establish any correlations, classical or quantum. However, the experimenters may communicate any information they see fit via a classical channel (carrier pigeon, smoke signals, e-mail). Any number of LO and CC rounds may be performed. In this manner one may set-up arbitrary

classical correlations. However, since all information exchanged between the separated parties at any point was intrinsically classical, LOCC cannot create genuine quantum correlations.

Two quantum states of a composite system are then said to be *stochastically* LOCC (SLOCC) equivalent if and only if they may be probabilistically interrelated using LOCC. Since LOCC cannot create entanglement, two SLOCC-equivalent states must possess the same “amount” of entanglement. For more details, see [38, 39] and Refs. therein.

Let us make this a little more precise by focusing on the specific case of multi-qubit systems. What is a qubit? Quantum information can live in a quantum mechanical superposition. Hence, the qubit is a quantum superposition of the classical binary digits “0” and “1”. The particular physical realisation (there are many: photon polarisations, quantum dots, trapped ions, mode splitters, to name but a few) of the qubit is not important, any two state quantum system will do. Hence, qubits are simply regarded abstractly as elements of the 2-dimensional Hilbert space \mathbb{C}^2 , equipped with the conventional norm, where the two basis states are labelled $|0\rangle$ and $|1\rangle$. An n -qubit bit string $|\Psi\rangle$ lives in the n -fold tensor product of \mathbb{C}^2 :

$$\begin{aligned} |\Psi\rangle &= a_{A_1 \dots A_n} |A_1\rangle \otimes |A_2\rangle \otimes \dots \otimes |A_n\rangle \\ &= a_{A_1 \dots A_n} |A_1 A_2 \dots A_n\rangle, \end{aligned} \quad (2.1)$$

where $a_{A_1 \dots A_n} \in \mathbb{C}$ and we sum over $A_1, \dots, A_n = 0, 1$. In [9] it was argued that two states of an n -qubit system are SLOCC-equivalent if and only if they are related by $[\text{SL}(2, \mathbb{C})]^{\otimes n}$, under which $a_{A_1 \dots A_n}$ transforms as the fundamental $(\mathbf{2}, \mathbf{2}, \dots, \mathbf{2})$ representation. In this respect, $[\text{SL}(2, \mathbb{C})]^{\otimes n}$ may be usefully thought of as the “gauge” group of n -qubit entanglement. Hence, the space of physically distinct n -qubit entanglement classes (or orbits) is given by

$$\frac{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}{\text{SL}_1(2, \mathbb{C}) \times \text{SL}_2(2, \mathbb{C}) \times \dots \times \text{SL}_n(2, \mathbb{C})}. \quad (2.2)$$

When classifying entanglement, it is this space we wish to understand.

This very quickly becomes a difficult task. Although two and three qubit entanglement is well-understood (see *e.g.* [9]), the literature on four qubits can be confusing and seemingly contradictory, as illustrated in Table 1. This is due in part to genuine

Table 1: Various results on four-qubit entanglement.

Paradigm	Author	Year	Ref	result mod perms	result incl. perms
classes	Wallach	2004	[40]	?	90
	Lamata et al	2006	[41]	8 genuine, 5 degenerate	16 genuine, 18 degenerate
	Cao et al	2007	[42]	8 genuine, 4 degenerate	8 genuine, 15 degenerate
	Li et al	2007	[43]	?	≥ 31 genuine, 18 degenerate
	Akhtarshenas et al	2010	[44]	?	11 genuine, 6 degenerate
	Bunty et al	2010	[45]	21 genuine, 5 degenerate	64 genuine, 18 degenerate
families	Verstraete et al	2002	[30]	9	?
	Cherental et al	2007	[31]	9	?
	String theory	2010	[24]	9	31

calculational disagreements, but in part to the use of distinct (but in principle consistent and complementary) perspectives on the criteria for classification.

On the one hand, there is the “covariant” approach which distinguishes the SLOCC orbits by the vanishing or not of $[\text{SL}(2, \mathbb{C})]^{\otimes n}$ covariants/invariants. This philosophy is adopted for the 3-qubit case in [9, 46], for example, where it was shown that three qubits can be tripartite entangled in two inequivalent ways, denoted W and GHZ (Greenberger-Horne-Zeilinger). The analogous 4-qubit case was treated, with partial results, in [47]. Several new systems, in addition to the 4-qubit example, have been studied using the covariant framework in interesting recent work employing algebraic invariants of linear maps [45, 48].

On the other hand, there is the “normal form” approach which considers “families” of orbits. An arbitrary state may be transformed into one of a finite number of normal forms. If the normal form depends on some of the algebraically independent SLOCC invariants it constitutes a family of orbits parametrised by these invariants. On the other hand, a parameter-independent family contains a single orbit. This philosophy is adopted for the 4-qubit case $|\Psi\rangle = a_{ABCD}|ABCD\rangle$ in [30, 31]. Up to permutation of the four qubits, these authors found 6 parameter-dependent families called G_{abcd} , L_{abc2} , L_{a2b2} , $L_{a203\oplus\bar{1}}$, L_{ab3} , L_{a4} and 3 parameter-independent families called $L_{03\oplus\bar{1}03\oplus\bar{1}}$, $L_{05\oplus\bar{3}}$, $L_{07\oplus\bar{1}}$. For example, a family of orbits parametrised by all four of the algebraically independent SLOCC invariants is given by the normal form G_{abcd} :

$$\begin{aligned} &\frac{(a+d)}{2}(|0000\rangle + |1111\rangle) + \frac{(a-d)}{2}(|0011\rangle + |1100\rangle) \\ &+ \frac{(b+c)}{2}(|0101\rangle + |1010\rangle) + \frac{(b-c)}{2}(|1001\rangle + |0110\rangle). \end{aligned} \quad (2.3)$$

To illustrate the difference between these two approaches, consider the separable EPR-EPR state $(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$. Since this is obtained by setting $b = c = d = 0$ in (2.3), it belongs to the G_{abcd} family, whereas in the covariant approach it forms its own class. Similarly, a totally separable A - B - C - D state, such as $|0000\rangle$, for which all covariants/invariants vanish, belongs to the family L_{abc_2} , which however also contains genuine four-way entangled states. These interpretational differences were also noted in [41].

As we shall see, our BH perspective lends itself naturally to the “normal form” framework.

3 Black Holes and Nilpotent Orbits

3.1 Time-like Reduction and Stationary Black Holes

We consider $D = 4$ supergravity theories coupled to n Abelian gauge potentials, in which the scalar fields coordinatise a symmetric coset $M_4 = G_4/H_4$, where G_4 is the global U-duality group¹ and H_4 is its maximal compact subgroup. In this paper we will consider both the $\mathcal{N} = 2$ STU supergravity coupled to three vector multiplets, for which $n = 4$, and the full $\mathcal{N} = 8$ theory, for which $n = 28$. Schematically, we have an action of the form

$$S = \int d^4x \sqrt{-g} \left[R_4 - \partial_\mu \phi_i \partial^\mu \phi_j \gamma_4^{ij} - \mu_{IJ} F^I \wedge \star F^J + \nu_{IJ} F^I \wedge F^J \right] \quad (3.1)$$

where R_4 is the Ricci scalar, ϕ^i are the scalar fields (coordinates in M_4), γ_4^{ij} is the M_4 metric and F^I are the n field strengths of the n Abelian gauge vectors. We are going to use a spherically symmetric static *Ansatz* of the form

$$ds^2 = -e^U dt^2 + e^{-U} (dr^2 + r^2 d\Omega^2) \quad (3.2)$$

to describe our BH background. If we were to compactify one of the space like directions, we would end up with a $D = 3$ theory of spacetime with a scalar manifold given by $M_3 = G_3/H_3$ where G_3 is the $D = 3$ duality group and H_3 is its maximal compact subgroup. Instead (for reasons that will become clear), we perform a time-like reduction to a $D = 3$ space (not spacetime) Σ_3 . In $D = 3$ all vectors can be dualised to scalars, such that after dualisation one ends up with a non-linear sigma model coupled to Euclidean gravity, *i.e.* an action of the form [50]

$$S = \int d^3x \sqrt{h} \left[\frac{1}{2} R_3 - \frac{1}{2} h^{ab} \partial_a \phi^i \partial_b \phi^j \gamma_{3ij} \right] \quad (3.3)$$

where h_{ab} is the (Euclidean) $D = 3$ metric of Σ_3 , R_3 is the Ricci scalar, ϕ^i are $D = 3$ scalars coordinatizing M_3^* (which is now a pseudo-Riemannian symmetric space $M_3^* = G_3/H_3^*$, where H_3^* is a suitable non-compact form of H_3), and γ_{3ij} is the M_3^* metric. One may wonder if this procedure is well defined and what the pay-off might be. Luckily, as explained in [50], as long as one considers stationary BHs with a well defined global time-like Killing vector (in the original $D = 4$) the mentioned procedure is well defined.

The equations of motion are

$$R_{ab} = \gamma_{3ij} \partial_a \phi^i \partial_b \phi^j; \quad (3.4)$$

$$D^\alpha \partial_\alpha \phi^i = 0. \quad (3.5)$$

With a judicious coordinate change to $\tau = r^{-1}$, the equations of motion for the scalars and gravity decouple, and reduce to geodesics on M_3^* that are parametrised by τ , given by

$$\frac{d^2 \phi^i}{d\tau^2} + \Gamma_{jk}^i \frac{d\phi^j}{d\tau} \frac{d\phi^k}{d\tau} = 0. \quad (3.6)$$

Physically, integrating out one of these geodesics corresponds to integrating the BH solution from $r = \infty$ to $r = 0$ at the horizon. These geodesics can be calculated from a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} \gamma_{3ij} \dot{\phi}^i \dot{\phi}^j \quad (3.7)$$

where the dots denote differentiation with respect to τ . The differential geometry of symmetric manifolds can thus be exploited in order to re-express Lagrangian (3.7) in terms of Lie algebra elements. Actually, the procedure under consideration moved the time coordinate (and the g_{tt} component of the metric) into the pseudo-Riemannian scalar manifold M_3^* itself.

¹For a recent review of the general theory of duality rotations in four-dimensional supergravity theories see *e.g.* [49].

The pay-off from this procedure is that the differential geometry tools associated with symmetric manifolds can be used to study properties of the BH solutions associated with the g_{tt} component of the original $D = 4$ metric (3.2). In particular, simple requirements, such as regularity and the type of geodesic curves, allow one to select stationary and extremal BHs. The Hamiltonian constraint is

$$\gamma_{3ij}\dot{\phi}^i\dot{\phi}^j = v^2. \quad (3.8)$$

Given that M_3^* is pseudo-Riemannian, the geodesics may be time-like, light-like or space-like according as the solution is non-extremal, extremal or over-extremal (unphysical).

One uses a coset representative $L \in G/H^*$, which transforms globally under G and locally under H^* as

$$L \rightarrow gL \quad g \in G; \quad (3.9)$$

$$L \rightarrow Lh \quad h \in H^*. \quad (3.10)$$

The vielbein and connection one-forms may be found in the Maurer-Cartan formula $L^{-1}dL \in \mathfrak{g}$

$$L^{-1}dL = d\phi^i V_i^A T_A \quad (3.11)$$

where the $T_A \in \mathfrak{g}$ are in the solvable (*i.e.* upper-triangular) parametrisation. One then further defines the symmetric matrix $M = L\eta L^T$ such that M transforms under global G in the adjoint as $M \rightarrow gMg^{-1}$ and is inert under local H^* (intuitively, M can be thought as a point on the manifold G/H^*). Thus, the following result is achieved [50]:

$$S = \int \frac{1}{2} \gamma_{ij} \dot{\phi}^i \dot{\phi}^j = \int \frac{1}{2} \gamma_{ij} V_A^i V_B^j \dot{\phi}^A \dot{\phi}^B = \int \frac{1}{2} \eta_{AB} \dot{\phi}^A \dot{\phi}^B \quad (3.12)$$

$$= \int \frac{1}{2} \text{tr}(T_A T_B) \dot{\phi}^A \dot{\phi}^B = \int \frac{1}{8} \text{tr}(\dot{M} \dot{M}^{-1}), \quad (3.13)$$

where the last line takes about half a page of calculation to manipulate the $\phi_A T^A$ into M . From here, the equation of motion is clearly seen as

$$\frac{d}{d\tau} \left[M^{-1} \frac{d}{d\tau} M \right], \quad (3.14)$$

and the solution is given simply by

$$M(\tau) \equiv M(\phi^i(\tau)) = M(0) \exp 2Q\tau. \quad (3.15)$$

where $Q \in \mathfrak{g}$ is an algebra element that will become central to the whole picture explained below.

Theorem 6.4 of [51] states that *any* static spherically symmetric BHs is G_3 -equivalent to the Schwarzschild one in which the only scalar turned on is $g_{tt} = \exp U$. Thus, one can study BH solutions by analyzing the orbits of M under G_3 . In turns out that [23, 52–54], in the adjoint, the Lie algebra valued matrix of $D = 3$ Noether charges Q satisfies

$$Q^5 = 5v^2 Q^3 - 4v^2 Q^2, \quad (3.16)$$

while in the fundamental it holds that

$$Q^3 = v^2 Q \quad (3.17)$$

where v^2 is the geodesic parameter. Therefore, for light-like geodesics (where $v^2 = 0$), corresponding to extremal BHs, one obtains that $Q^3 = 0$ is nilpotent. From (3.15), this implies that M terminates at

$$M = \left[\mathbb{I} + \tau Q + \frac{1}{2} \tau^2 Q^2 \right], \quad (3.18)$$

and that the problem of classifying extremal BH solutions reduces to the problem of classifying orbits of nilpotent $Q \in \mathfrak{g}$ or, in other words, the nilpotent orbits of G_3 given are in one-to-one correspondence with the extremal BHs of the original $D = 4$ theory.

By specializing the above reasoning to the STU model, one gets²

$$\frac{G_4}{H_4} = \frac{\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})}{\text{SO}(2) \times \text{SO}(2) \times \text{SO}(2)}, \text{rank} = 3; \quad (3.19)$$

$$\frac{G_3}{H_3^*} = \frac{\text{SO}(4, 4)}{\text{SO}(2, 2) \times \text{SO}(2, 2)}, \text{rank} = 4. \quad (3.20)$$

²The *rank* of a globally symmetric space is defined as the maximal dimension (in \mathbb{R}) of a *flat* (*i.e.* with vanishing Riemann tensor), *totally geodesic* submanifold of such a space (see *e.g.* §6, page 209 of [55]).

Whereas for the maximal $\mathcal{N} = 8$ theory it holds

$$\frac{G_4}{H_4} = \frac{E_{7(7)}}{\text{SU}(8)}, \text{rank} = 7; \quad (3.21)$$

$$\frac{G_3}{H_3^*} = \frac{E_{8(8)}}{\text{SO}^*(16)}, \text{rank} = 8. \quad (3.22)$$

In $\mathcal{N} = 2$ theories, the relation between the special Kähler (see *e.g.* [56, 57] and Refs. therein) symmetric coset (3.19) and the para-quaternionic symmetric coset (3.19) is mathematically expressed through the “*-version” of the c -map [58] (see also [52], and Refs. therein).

In the case of the STU model, thanks to the Kostant-Sekiguchi correspondence (see next Subsection), the nilpotent orbits of G_3/H_3^* are diffeomorphic to the complex nilpotent orbits of $[\text{SL}(2, \mathbb{C})]^4$ on its fundamental, which happens to be the classification of four qubits, see the treatment in Sec. 4.

3.2 The Kostant-Sekiguchi Theorem

Consider a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with Cartan decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$. It holds that

$$[\mathfrak{h}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}] \subseteq \mathfrak{k}_{\mathbb{C}}, \quad [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}] \subseteq \mathfrak{h}_{\mathbb{C}}, \quad [\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}] \subseteq \mathfrak{h}_{\mathbb{C}}, \quad (3.23)$$

(*i.e.* $\mathfrak{h}_{\mathbb{C}}$ is a sub-algebra of $\mathfrak{g}_{\mathbb{C}}$). The $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ algebras, have corresponding complex Lie groups, $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$, that have a natural adjoint action on their respective algebras, given by

$$\mathfrak{a} \rightarrow g\mathfrak{a}g^{-1} \quad (3.24)$$

where $\mathfrak{a} \in \mathfrak{g}$ and $g \in G_{\mathbb{C}}$. Consider further the real forms of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ given respectively by $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}$ and their respective real groups $G_{\mathbb{R}}$ and $H_{\mathbb{R}}$. Then the Kostant-Sekiguchi theorem [29] states that the adjoint orbits of $G_{\mathbb{R}}$ on elements of $\mathfrak{g}_{\mathbb{R}}$ that are nilpotent are diffeomorphic to the nilpotent fundamental orbits of $H_{\mathbb{C}}$ on $\mathfrak{k}_{\mathbb{C}}$, *i.e.*

$$\frac{\mathfrak{N} \cap \mathfrak{g}_{\mathbb{R}}}{G_{\mathbb{R}}} \leftrightarrow \frac{\mathfrak{N} \cap \mathfrak{k}_{\mathbb{C}}}{H_{\mathbb{C}}}. \quad (3.25)$$

where \mathfrak{N} is the variety of nilpotent elements.

For the STU model, we pick

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(8)_{\mathbb{C}} = \mathfrak{so}(4)_{\mathbb{C}} + \mathfrak{so}(4)_{\mathbb{C}} + (\mathbf{4}, \mathbf{4}) \quad (3.26)$$

$$= \mathfrak{sl}(2)_{\mathbb{C}} + \mathfrak{sl}(2)_{\mathbb{C}} + \mathfrak{sl}(2)_{\mathbb{C}} + \mathfrak{sl}(2)_{\mathbb{C}} + (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (3.27)$$

$$= \mathfrak{h}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}, \quad (3.28)$$

therefore, we have $H_{\mathbb{C}} = \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ and $\mathfrak{k}_{\mathbb{C}} = (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})_{\mathbb{C}}$. We choose $G_{\mathbb{R}} = \text{SO}_0(4, 4)$, where the 0 subscript denotes the identity-connected component, and pick the non-compact version of the maximal compact subgroup $H_{\mathbb{R}}^* = \text{SO}(2, 2) \times \text{SO}(2, 2)$ ³. In this way the Kostant-Sekiguchi correspondence tells us that

$$\text{Nilpotent} \frac{\mathfrak{so}(4, 4)}{\text{SO}(4, 4)} \sim \text{Nilpotent} \frac{(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})_{\mathbb{C}}}{\text{SL}(2, \mathbb{C})^4}. \quad (3.29)$$

while for the $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 2$ exceptional model, we choose

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{e}(8)_{\mathbb{C}} = \mathfrak{so}^*(16)_{\mathbb{C}} + (\mathbf{128}) \quad (3.30)$$

$$= \mathfrak{h}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}. \quad (3.31)$$

However, as it will be discussed in Sects. 5 and 6, the QIT interpretations of both the $\mathcal{N} = 8$ and the exceptional $\mathcal{N} = 2$ theories are not amenable to the application of the Kostant-Sekiguchi correspondence.

³The Kostant-Sekiguchi theorem applies to non-compact $H_{\mathbb{R}}^*$ as well. The details are in the first appendix of [53]

4 The STU model and the Entanglement of Four Qubits

4.1 Summary

Here we briefly summarise the relationship between the classes of STU BH solutions and the entanglement classes of four qubits. In the following Section we provide a more comprehensive analysis.

The STU model [3–5, 59] is a particular model of $\mathcal{N} = 2$ supergravity coupled to three vector multiplets. It has three complex scalars denoted S, T and U , which parameterize the symmetric coset space (3.19).

The static, asymptotically flat, spherically symmetric, extremal⁴ BH solutions of the STU model are characterized by a maximum of 8 charges (four electric and four magnetic), namely 1 + 3 from the gravity and vector multiplets respectively, plus their magnetic duals. Hence, the Bekenstein-Hawking entropy [60, 61] is a function of the 8 charges. Through scalar-dressing, these charges can be grouped into the $\mathcal{N} = 2$ central charge z and three “matter charges”. Depending on the values of the charges, the extremal BHs are divided into “small” or “large”, according as their Bekenstein-Hawking [60, 61] entropy is zero or not. The “small” ones are termed *lightlike*, *critical* or *doubly-critical*, depending on the minimal number (under U -duality) of representative electric or magnetic charges, which is respectively 3, 2 or 1. One subtlety is that some extremal cases, termed “extremal”, cannot be obtained as limits of non-extremal BHs (see Sect. 4.4).

Performing a time-like reduction to $D = 3$, one obtains the pseudo-Riemannian para-quaternionic symmetric coset (3.20). Hence, the extremal solutions are classified by the nilpotent orbits of $SO(4, 4)$ acting on its adjoint representation **28**. Here we consider the finer classification, obtained from the nilpotent orbits of $SO_0(4, 4)$. These orbits may be labeled by “signed” Young tableaux, often referred to as *ab*-diagrams in the mathematics literature (see *e.g.* [62], and Refs. therein). Each signed Young tableau, as listed in Table 2, actually corresponds to a single nilpotent $O(4, 4)$ orbit, of which the $SO_0(4, 4)$ nilpotent orbits are the connected components. Since $O(4, 4)$ has four components, for each nilpotent $O(4, 4)$ orbit there may be either 1, 2 or 4 nilpotent $SO_0(4, 4)$ orbits. This number is also determined by the corresponding signed Young tableau. If the middle sign of every odd length row is “−” (“+”) there are 2 orbits and we label the diagram to its left (right) with a I or a II . If it is none of these, it is said to be stable and there is only one orbit. The signed Young tableaux together with their labellings, as listed in Table 2, give a total of 31 nilpotent $SO_0(4, 4)$ orbits [24]. The matching of the extremal classes to the nilpotent orbits is given in Table 2 [24], and it is discussed in detail in Sects. 4.2–4.4. We also supply the complete list of the associated cosets in Table 2, some of which may be found in [53].

In order to relate the extremal BH solutions to the entanglement classes of four qubits, we invoke the aforementioned Kostant-Sekiguchi theorem [28, 29]. Noting the convenient isomorphism $SO(2, 2) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, the scalar manifold G_3/H_3^* of the time-like reduced STU model may be rewritten as $SO(4, 4)/[SL(2, \mathbb{R})]^4$, which yields the Cartan decomposition

$$\mathfrak{so}(4, 4) \cong [\mathfrak{sl}(2, \mathbb{R})]^4 \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}). \quad (4.1)$$

The relevance of (4.1) to four qubits was pointed out in [19] and recently spelled out more clearly by Levay [23], who relates four qubits to $D = 4$ STU BHs. The 16 independent components are given by the 4 + 4 electromagnetic charges, the NUT charge, the mass and three complex scalars of the STU model. By applying the Kostant-Sekiguchi correspondence to the Cartan decomposition (4.1), one can state that the nilpotent orbits of $SO_0(4, 4)$ acting on its adjoint representation are in one-to-one correspondence with the nilpotent orbits of $[SL(2, \mathbb{C})]^4$ acting on its fundamental $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$ representation and, hence, with the classification of 4-qubit entanglement. Note furthermore that it is the complex qubits that appear automatically, thereby relaxing the restriction to real qubits (sometimes called rebits) that featured in earlier versions of the BH/qubit correspondence.

It follows that there are 31 nilpotent orbits for four qubits under SLOCC [24]. For each nilpotent orbit there is precisely one family of SLOCC orbits since each family contains one nilpotent orbit on setting all invariants to zero. The nilpotent orbits and their associated families are summarized in Table 2 [24], which is split into upper and lower sections according as the nilpotent orbits belong to parameter-dependent or parameter-independent families.

If one allows for the permutation of the four qubits, the connected components of each $O(4, 4)$ orbit are re-identified reducing the count to 17. Moreover, these 17 are further grouped under this permutation symmetry into just nine nilpotent orbits. It is not difficult to show that these nine cosets match the nine families of [30, 31], as listed in the final column of Table 2 (provided we adopt the version of L_{ab_3} presented in [31] rather than in [30]). For example, the state representative $|0111\rangle + |0000\rangle$ of the family $L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$ is left invariant by the $[SO(2, \mathbb{C})]^2 \times \mathbb{C}$ subgroup, where $[SO(2, \mathbb{C})]^2$ is the stabilizer of the three-qubit GHZ state [46]. In contrast, the four-way entangled family $L_{0_7 \oplus \bar{1}}$, which is the “principal” nilpotent orbit [29], is not left invariant by any subgroup. Note that the total of 31 does not follow trivially by permuting the qubits in these nine. Naive permutation produces far more than 31 candidates, which then have to be reduced to SLOCC inequivalent families.

⁴BHs are divided into extremal and non-extremal, according as their Hawking temperature is zero or not. The corresponding Noether orbits in $D = 3$ are nilpotent or semisimple, respectively.

There is a satisfying consistency of this process with respect to the covariant approach (which, as mentioned, is the other criterion for classification). For example, the covariant classification has four biseparable classes A -GHZ, B -GHZ, C -GHZ and D -GHZ which are then identified as a single class under the permutation symmetry. These four classes are in fact the four nilpotent orbits corresponding to the families $L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$ in Table 2, which are also identified as a single nilpotent orbit under permutations. Similarly, each of the four A -W classes is a nilpotent orbit belonging to one of the four families labeled $L_{a_2 0_3 \oplus \bar{1}}$ which are again identified under permutations. A less trivial example is given by the six A - B -EPR classes of the covariant classification. These all lie in the single family $L_{a_2 b_2}$ of [30], which is defined up to permutation. Consulting Table 2 we see that, when not allowing permutations, this family splits into six pieces, each containing one of the six A - B -EPR classes. Finally, the single totally separable class A - B - C - D is the single nilpotent orbit inside the single family L_{abc_2} , which maps into itself under permutations.

4.2 “Large” (i.e. Attractor) Extremal $D = 4$ STU Black Holes

The five nilpotent orbits of $SO_0(4, 4)$ of $\dim_{\mathbb{R}} = 18$ [29] (which correspond to $L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$ and L_{ab_3} four qubits entanglement families [24]) are related to extremal “large” (and thus attractor) STU $D = 4$ black holes (BHs). As discussed *e.g.* in App. A.3 of [53], they are characterized by

$$A^5 = 0; \quad (4.2)$$

$$V^3 = S^3 = C^3 = 0, \quad (4.3)$$

where $A \equiv \mathbf{28}$, $S \equiv \mathbf{8}_s$, $V \equiv \mathbf{8}_v$ and $C \equiv \mathbf{8}_c$ respectively denote the adjoint, vector, spinor and conjugate spinor irreps. of the $D = 3$ U-duality group $G_{3,STU} = SO_0(4, 4)$. Thus, the *triality* symmetry exhibited by the $\mathcal{N} = 2$, $D = 4$ STU model [3, 4] can be traced back to the triality of irreps. $\mathbf{8}$ ’s of $G_{3,STU}$ itself⁵. Conditions (4.2) and (4.3) are exactly the ones requested for extremal “large” (and thus attractor) STU $D = 4$ BHs (see Eq. (4.27) below).

For use in the subsequent treatment, let us introduce the following maps of cyclical index permutations: the *triality* τ (pertaining to $D = 4$; \mathbb{I} denotes the identity throughout)

$$\tau : 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 2; \quad \tau^3 = \mathbb{I}; \quad (4.4)$$

and the *quaternionality* π (pertaining to $D = 3$)

$$\pi : 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1; \quad \pi^4 = \mathbb{I}. \quad (4.5)$$

As evident from the treatment given below, τ does commute with $D = 4$ supersymmetry, whereas π does or does not, depending on the case.

4.2.1 STU Parametrization of $\mathcal{N} = 8$, $D = 4$ Supergravity

The supergravity interpretation of the $SO_0(4, 4)$ -nilpotent orbits of $\dim_{\mathbb{R}} \leq 18$ considered below is based on the so-called “ STU parametrization” of $\mathcal{N} = 8$, $D = 4$ supergravity, discussed *e.g.* in [64]. This amounts to identifying the $\mathcal{N} = 2$ central charge and the three STU matter charges with the four skew-eigenvalues z_i ($i = 1, \dots, 4$ throughout) of the $\mathcal{N} = 8$ central charge matrix as follows [64, 65]

$$Z \equiv z_1; \quad \sqrt{g^{s\bar{s}}} \bar{D}_{\bar{s}} \bar{Z} \equiv iz_2; \quad \sqrt{g^{t\bar{t}}} \bar{D}_{\bar{t}} \bar{Z} \equiv iz_3; \quad \sqrt{g^{u\bar{u}}} \bar{D}_{\bar{u}} \bar{Z} \equiv iz_4. \quad (4.6)$$

Thus, the effective BH potential V_{BH} , its criticality conditions (*alias* the Attractor Eqs.) and the quartic invariant \mathcal{I}_4 of $\mathcal{N} = 2$, $D = 4$ STU model can be traded for the ones pertaining to maximal supergravity, respectively reading [64, 66, 67]:

$$V_{BH} = \sum_i |z_i|^4; \quad (4.7)$$

$$\partial_\phi V_{BH} = 0 \Leftrightarrow z_i z_j + \bar{z}_k \bar{z}_l = 0, \quad \forall i \neq j \neq k \neq l; \quad (4.8)$$

$$\mathcal{I}_4 = \sum_i |z_i|^4 - 2 \sum_{i < j} |z_i|^2 |z_j|^2 + 4 \left(\prod_i z_i + \prod_i \bar{z}_i \right), \quad (4.9)$$

⁵In general, the relevant non-compact subalgebra $\mathfrak{h}_3^* = \mathfrak{g}_4$ for the application of the Kostant-Sekiguchi Theorem ([29], and Refs. therein) to the issue of extremal BHs is the unique non-compact form of \mathfrak{h}_3 ($\mathfrak{h}_3 \oplus \mathfrak{su}(2)$ being the maximal compact subalgebra symmetrically embedded into \mathfrak{g}_3) such that it is embedded *maximally* (through a commuting $\mathfrak{sl}(2, \mathbb{R})$ algebra) and *symmetrically* into \mathfrak{g}_3 itself. At geometric level, \mathfrak{h}_3^* is selected through the c^* -map, which is the generalization, pertaining to timelike $D = 4 \rightarrow D = 3$ reduction, of the c -map [58] (for a review, and a list of Refs., see *e.g.* [52]). Thus, in the STU case ($\mathfrak{g}_{3,STU} = \mathfrak{so}(4, 4)$, $\mathfrak{h}_{3,STU}^* = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$), the “black hole/qubit correspondence” [24] exploited through the Kostant-Sekiguchi Theorem (for identity connected components), enjoys a geometrical interpretation in terms of c^* -map (*e.g.* see explicit treatment of c^* -map of STU model in [52, 63]).

Table 2: Each black hole nilpotent $SO_0(4, 4)$ orbit corresponds to a 4-qubit nilpotent $[SL(2, \mathbb{C})]^4$ orbit [24]

STU black holes					Four qubits	
description	Young tableaux	$SO_0(4, 4)$ coset	$\dim_{\mathbb{R}}$	$[SL(2, \mathbb{C})]^4$ coset	nilpotent rep	family
trivial	trivial	$\frac{SO_0(4,4)}{SO_0(4,4)}$	1	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C})]^4}$	0	G_{abcd}
doubly-critical $\frac{1}{2}$ BPS		$\frac{SO_0(4,4)}{[SL(2, \mathbb{R}) \times SO(2, 2, \mathbb{R})] \ltimes [(\mathbf{2}, \mathbf{4})^{(1)} \oplus \mathbf{1}^{(2)}]}$	10	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C})]^3 \ltimes \mathbb{C}^4}$	$ 0110\rangle$	L_{abc_2}
critical, $\frac{1}{2}$ BPS and non-BPS		$\frac{SO_0(4,4)}{SO(3, 2; \mathbb{R}) \ltimes [(\mathbf{5} \oplus \mathbf{1})^{(2)}]}$				
		$\frac{SO_0(4,4)}{SO(2, 3; \mathbb{R}) \ltimes [(\mathbf{5} \oplus \mathbf{1})^{(2)}]}$	12	$\frac{[SL(2, \mathbb{C})]^4}{[SO(3, \mathbb{C}) \times \mathbb{C}] \times [SO(2, \mathbb{C}) \ltimes \mathbb{C}]}$	$ 0110\rangle + 0011\rangle$	$L_{a_2 b_2}$
	$\left(\begin{array}{c} \text{Young diagram 1} \\ \text{Young diagram 2} \end{array} \right) \quad I, II$	$\frac{SO_0(4,4)}{Sp(4, \mathbb{R}) \ltimes [(\mathbf{5} \oplus \mathbf{1})^{(2)}]}$				
lightlike $\frac{1}{2}$ BPS and non-BPS	$\left(\begin{array}{c} \text{Young diagram 1} \\ \text{Young diagram 2} \end{array} \right) \quad I, II$	$\frac{SO_0(4,4)}{SL(2, \mathbb{R}) \ltimes [(2 \times \mathbf{2})^{(1)} \oplus (3 \times \mathbf{1})^{(2)} \oplus \mathbf{2}^{(3)}]}$	16	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C}) \ltimes \mathbb{C}] \times \mathbb{C}^2}$	$ 0110\rangle + 0101\rangle + 0011\rangle$	$L_{a_2 0_3 \oplus \bar{1}}$
large non-BPS $z_H \neq 0$		$\frac{SO_0(4,4)}{SO(1, 1, 1, \mathbb{R}) \times SO(1, 1, \mathbb{R}) \ltimes [((\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}))^{(2)} \oplus \mathbf{1}^{(4)}]}$	18	$\frac{[SL(2, \mathbb{C})]^4}{\mathbb{C}^3}$	$\frac{i}{\sqrt{2}}(0001\rangle + 0010\rangle - 0111\rangle - 1011\rangle)$	L_{ab_3}
“extremal”		$\frac{SO_0(4,4)}{SO(2, 1; \mathbb{R}) \ltimes [\mathbf{1}^{(2)} \oplus \mathbf{3}^{(4)} \oplus \mathbf{1}^{(6)}]}$				
		$\frac{SO_0(4,4)}{SO(1, 2; \mathbb{R}) \ltimes [\mathbf{1}^{(2)} \oplus \mathbf{3}^{(4)} \oplus \mathbf{1}^{(6)}]}$	20	$\frac{[SL(2, \mathbb{C})]^4}{SO(2, \mathbb{C}) \times \mathbb{C}}$	$i 0001\rangle + 0110\rangle - i 1011\rangle$	L_{a_4}
	$\left(\begin{array}{c} \text{Young diagram 1} \\ \text{Young diagram 2} \end{array} \right) \quad I, II$	$\frac{SO_0(4,4)}{Sp(2, \mathbb{R}) \ltimes [\mathbf{1}^{(2)} \oplus \mathbf{3}^{(4)} \oplus \mathbf{1}^{(6)}]}$				
large $\frac{1}{2}$ BPS and non-BPS $z_H = 0$	$\left(\begin{array}{c} \text{Young diagram 1} \\ \text{Young diagram 2} \end{array} \right) \quad I, II$	$\frac{SO_0(4,4)}{SO(2, \mathbb{R}) \times SO(2, \mathbb{R}) \ltimes [((\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}))^{(2)} \oplus \mathbf{1}^{(4)}]}$	18	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C})]^2 \times \mathbb{C}}$	$ 0000\rangle + 0111\rangle$	$L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$
“extremal”	$\left(\begin{array}{c} \text{Young diagram 1} \\ \text{Young diagram 2} \end{array} \right) \quad I, II$	$\frac{SO_0(4,4)}{\mathbb{R}^{3(2)} \oplus \mathbb{R}^{1(4)} \oplus \mathbb{R}^{2(6)}}$	22	$\frac{[SL(2, \mathbb{C})]^4}{\mathbb{C}}$	$ 0000\rangle + 0101\rangle + 1000\rangle + 1110\rangle$	$L_{0_5 \oplus 3}$
“extremal”	$\left(\begin{array}{c} \text{Young diagram 1} \\ \text{Young diagram 2} \end{array} \right) \quad I, II$	$\frac{SO_0(4,4)}{\mathbb{R}^{(2)} \oplus \mathbb{R}^{2(6)} \oplus \mathbb{R}^{(10)}}$	24	$\frac{[SL(2, \mathbb{C})]^4}{\mathbb{I}}$	$ 0000\rangle + 1011\rangle + 1101\rangle + 1110\rangle$	$L_{0_7 \oplus \bar{1}}$

where the notation “ $i \neq j \neq k \neq l$ ” means all different indices throughout.

In the subsequent treatment, we will also consider $\mathcal{N} = 4$, $D = 4$ supergravity, and we will use the ($\mathcal{N} = 2$ *STU* analogue of the) $\mathcal{N} = 4$, $D = 4$ normal frame adopted in [68], which is more convenient to unravel the relations to $\mathcal{N} = 8$, $D = 4$ supergravity, and the corresponding *quaternionality* properties.

Due to maximal $\mathcal{N} = 8$ supersymmetry, note that (4.7), (4.8) and (4.9) are manifestly π -invariant, as it can be checked at a glance by recalling (4.5). By performing a suitable \mathcal{R} -symmetry $SU(8)$ -transformation, the Hua-Bloch-Messiah-Zumino theorem [69–71] allows one to set the phases of z_i ’s to be all equal, namely:

$$z_i \equiv |z_i| e^{i\frac{\varphi}{4}}, \quad \forall i, \quad \varphi \in [0, 8\pi). \quad (4.10)$$

This has been named “special normal frame” in [72]. It should also be pointed out that, out of (4.8), only some of them are independent up to π -transformations and complex conjugation, namely:

$$\begin{cases} z_1 z_2 + \overline{z_3 z_4} = 0; \\ z_1 z_3 + \overline{z_2 z_4} = 0. \end{cases} \quad (4.11)$$

4.2.2 (A, B, C, D) -GHZ Classes $\Leftrightarrow L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$: $\frac{1}{2}$ -BPS and Non-BPS $Z_H = 0$ “Large” BHs

The four nilpotent orbits corresponding to the family $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ are classes of bi-separable four qubit entanglement, namely *A*-GHZ, *B*-GHZ, *C*-GHZ and *D*-GHZ [24]. The corresponding Young tableaux are related through π , and they actually reduce to only one up to π -transformations.

It is convenient to set $i \in \mathbb{N} \bmod 4$ (i.e. $i + 4 \equiv i$). Then, $\mathcal{N} = 8$, $D = 4$ $\frac{1}{8}$ -BPS solutions to (4.11) are given by [64]

$$\forall i \begin{cases} z_i \neq 0; \\ z_{i+1} = z_{i+2} = z_{i+3} = 0, \end{cases} \quad (4.12)$$

with φ *undetermined* (thus, the non-vanishing z_i ’s are generally complex). It is evident that the four solutions (4.12) are related through π . They exhibit the maximal compact symmetry consistent with the charge orbit [73–76] (see also [77] for a treatment of “moduli spaces” of attractors)

$$\mathcal{O}_{\mathcal{N}=8, \frac{1}{8}\text{-BPS}, \text{large}} = \frac{E_{7(7)}}{E_{6(2)}}, \quad (4.13)$$

namely $SU(2) \times SU(6)$ (obtained through the symmetry enhancement at the BH event horizon), which is the *maximal compact subgroup* (*mcs*) of the stabilizer $E_{6(2)}$.

In the *STU* model, lower $\mathcal{N} = 2$ supersymmetry puts one of the four $\mathcal{N} = 8$ skew-eigenvalues, say z_1 (without loss of generality, up to re-labelling), on a *primus inter pares* status, corresponding to $\mathcal{N} = 2$ central charge. Thus, solutions (4.12) split into ($a = 2, 3, 4$ throughout)

$$\frac{1}{2}\text{-BPS} : \begin{cases} z_1 = 0; \\ z_a = 0 \quad \forall a; \end{cases} \quad (4.14)$$

$$\text{nBPS } Z_H = 0|_a : \begin{cases} z_1 = 0; \\ z_a \neq 0; \\ z_b = 0, \quad \forall b \neq a. \end{cases} \quad (4.15)$$

Note that the *STU* *triality* symmetry is implemented through τ (recall definition (4.4)). Thus, $\mathcal{N} = 8$ $\frac{1}{8}$ -BPS attractor solution (4.12) splits into:

- one τ -invariant (i.e. *triality*-invariant) $\mathcal{N} = 2$ $\frac{1}{2}$ -BPS attractor solution (4.14);
- three $\mathcal{N} = 2$ *STU* non-BPS $Z_H = 0$ solutions, related through τ [5]. After the analysis in App. AII of [78], the solutions with $a = 1$ and $a = \{2, 3\}$ would respectively correspond to class *II* and class *I* of non-BPS $Z_H = 0$ attractors. However, in the *STU* the corresponding charge orbits are *isomorphic* [78], because of the underlying *triality* symmetry, *cfr.* Eq. (4.17) below (see also [79]).

Furthermore, solutions (4.14)-(4.15) have different uplift properties to $\mathcal{N} = 4$, $D = 4$ supergravity (with $n_V = 6$ matter vector multiplets). In fact:

- (4.14) and (4.15) with $a = 1$ uplift to $\mathcal{N} = 4$ $\frac{1}{4}$ -BPS attractors;
- (4.15) with $a = 2, 3$ uplift to non-BPS attractors with vanishing horizon central charge ($Z_{AB}|_H = 0$) [80].

The resulting supersymmetry reduction scheme reads

$$\begin{array}{ccc}
\underline{\mathcal{N} = 8} : & \mathcal{O}_{\mathcal{N}=8, \frac{1}{8}\text{-BPS}, \text{large}} & \searrow \\
\underline{\mathcal{N} = 4} : & \mathcal{O}_{\mathcal{N}=4, n_V=6, \frac{1}{4}\text{-BPS}, \text{large}} & \mathcal{O}_{\mathcal{N}=4, n_V=6, n\text{BPS}, Z_{AB,H}=0, \text{large}} \\
& \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6,6)}{\text{SO}(2) \times \text{SO}(4,6)} & \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6,6)}{\text{SO}(2) \times \text{SO}(6,4)} \\
& \downarrow & \downarrow \\
\underline{\mathcal{N} = 2} : & \mathcal{O}_{\mathcal{N}=2, STU, \frac{1}{2}\text{-BPS}, \text{large}} & \mathcal{O}_{\mathcal{N}=2, STU, n\text{BPS}, Z_H=0, I, \text{large}} \\
& \updownarrow^* & \\
& \mathcal{O}_{\mathcal{N}=2, STU, n\text{BPS}, Z_H=0, II, \text{large}} &
\end{array} \tag{4.16}$$

“ \updownarrow^* ” indicates that the two orbits are related by the exchange $z_1 \longleftrightarrow z_2$, and [78] (see also [46])

$$\mathcal{O}_{\mathcal{N}=2, STU, \frac{1}{2}\text{-BPS}, \text{large}} \sim \mathcal{O}_{\mathcal{N}=2, STU, n\text{BPS}, Z_H=0, II, \text{large}} \sim \mathcal{O}_{\mathcal{N}=2, STU, n\text{BPS}, Z_H=0, I, \text{large}} = \frac{[\text{SL}(2, \mathbb{R})]^3}{[\text{U}(1)]^2}. \tag{4.17}$$

Notice that the exchange $z_1 \longleftrightarrow z_2$ of the two $\mathcal{N} = 4$ skew-eigenvalues implies a flip of the sign of the $\mathcal{N} = 2$ $H_{STU} = [\text{U}(1)]^3$ -invariant function

$$|Z|^2 - g^{s\bar{s}} |D_s Z|^2, \tag{4.18}$$

which in general allows one to discriminate between $\frac{1}{2}$ -BPS attractor and non-BPS $Z_H = 0$ of class *II* in the sequence of symmetric special Kähler geometries based on $\mathbb{R} \oplus \mathbf{\Gamma}_{1, n-1}$ (whose the *STU* model is the $n = 2$ element); see the discussion in App. AII of [78].

(4.16)-(4.17) correspond to the following chains of maximal symmetric embeddings⁶, respectively for the numerator and the stabilizer groups of the cosets:

$$E_{7(7)} \supsetneq \text{SL}(2, \mathbb{R}) \times \text{SO}(6, 6) \supsetneq [\text{SL}(2, \mathbb{R})]^3 \times \text{SO}(4, 4); \tag{4.19}$$

$$E_{6(2)} \supsetneq \text{SO}(2) \times \text{SO}(6, 4) \supsetneq [\text{SO}(2)]^2 \times \text{SO}(4, 4). \tag{4.20}$$

Note that the groups of the chain (4.20) are maximally and symmetrically embedded into the group of the chain (4.19) only through a factor group $\text{SO}(2)$.

As mentioned above, the correspondence of $\mathcal{N} = 4$ supergravity with the maximal theory is highlighted within the “democratic normal frame” recently introduced in [68], in which the \mathcal{R} -symmetry reduction in $D = 4$:

$$\begin{array}{ccc}
\mathcal{N} = 8 & \longrightarrow & \mathcal{N} = 4, n_V = 6 \\
\text{SU}(8) & \supsetneq & \text{U}(4) \times \text{SO}(6) \sim \text{U}(4) \times \text{SU}(4)
\end{array} \tag{4.21}$$

is fully manifest. In such a “democratic” normal frame, the two skew-eigenvalues z_1 and z_2 of the $\mathcal{N} = 4$ central charge are taken to be real non-negative through a suitable $\text{U}(4)$ -transformation, whereas the overall $\mathcal{N} = 8$ phase φ becomes, after a suitable $\text{SO}(6)$ -transformation, the overall phase in front of the unique two non-vanishing components $\tilde{\rho}_1 \equiv |z_3|$ and $\tilde{\rho}_2 \equiv |z_4|$ of $\mathcal{N} = 4$ matter charges’ vector Z_I ($I = 1, \dots, n_V = 6$) [68].

As mentioned, from the QIT perspective [24] $\mathcal{N} = 2$ supersymmetry singles out one qubit, say $A(\text{lice})$, on a *primus inter pares* status, because it is the (complexification of the) Ehlers $\text{SL}(2, \mathbb{R})$ determined by timelike $D = 4 \longrightarrow D = 3$ reduction (see e.g. [52] for a recent treatment and list of Refs.). Thus, solution (4.14) corresponds to an *A*-GHZ state, whereas solutions (4.15) correspond to *B*-GHZ, *C*-GHZ and *D*-GHZ states [23]. This is also consistent with the analysis of [31], characterizing $L_{0_3 \oplus \bar{1}^{0_3 \oplus \bar{1}}}$ as a distinguished family of bi-separable four qubit states.

The $\mathcal{N} = 2$ $D = 4$ *STU* interpretation given above is further confirmed by the fact that the non-translational part of the stabilizer of the nilpotent $\text{SO}_0(4, 4)$ -orbits under consideration, i.e. $[\text{SO}(2; \mathbb{R})]^2 \sim [\text{U}(1)]^2$, coincides⁷ with the stabilizer of the rank-4 GHZ orbits (see [78], as well as Table VI of [46]) (4.17).

⁶Unless otherwise noted, in the present investigation all group embeddings are maximal and symmetric.

⁷For a discussion of the relation between the nilpotent orbits of the $D = 3$ *U*-duality group G_3 and the charge orbits of the corresponding $D = 4$ *U*-duality group G_4 , see the end of Sec. 2.4 of [53].

The fact that the solutions (4.12) are related through π corresponds to four Young tableaux which are related through $D = 3$ permutation symmetry. Correspondingly, there exist four nilpotent $SO_0(4, 4)$ -orbits of dimension 18 (related to $L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$), which reduce to only one up to π -transformations. This is consistent with the fact that the $\frac{1}{2}$ -BPS “large” and both types (I and II) of non-BPS $Z_H = 0$ charge orbits of $\mathcal{N} = 2$, $D = 4$ STU model are *isomorphic*, as given by Eq. (4.17) [78].

4.2.3 4-way Entanglement in L_{ab_3} : Non-BPS $Z_H \neq 0$ “Large” BHs

The $SO_0(4, 4)$ -nilpotent orbit corresponding to the L_{ab_3} four qubit entanglement family is related to a Young tableaux which is invariant under $D = 3$ permutations [24]. Consistently, the corresponding solution to (4.11) is π -invariant [64]:

$$\begin{cases} z_i = \rho e^{i\frac{\varphi}{4}}, \rho \in \mathbb{R}_0^+, \forall i, \\ \varphi = \pi + 2k\pi, k \in \mathbb{Z}. \end{cases} \quad (4.22)$$

In $D = 4$, solution (4.22) is non-BPS in $\mathcal{N} = 8$, non-BPS with $Z_{AB}|_H \neq 0$ ($A, B = 1, \dots, 4$) in $\mathcal{N} = 4$ (with $n_V = 6$ matter vector multiplets), and non-BPS $Z_H \neq 0$ in $\mathcal{N} = 2$ STU model. It exhibits the maximal compact symmetry consistent with [73, 74]

$$\mathcal{O}_{\mathcal{N}=8, nBPS} = \frac{E_{7(7)}}{E_{6(6)}}, \quad (4.23)$$

namely $USp(8) = mcs(E_{6(6)})$ (obtained through the symmetry enhancement at the black hole event horizon).

As mentioned, solutions (4.22) uplift to non-BPS attractors with $Z_{AB}|_H \neq 0$ of $\mathcal{N} = 4$, $D = 4$, $n_V = 6$ supergravity [80]. The resulting supersymmetry reduction pattern reads

$$\begin{array}{ccc} \underline{\mathcal{N} = 8} : & \mathcal{O}_{\mathcal{N}=8, nBPS} & \\ & \downarrow & \\ & \mathcal{O}_{\mathcal{N}=4, n_V=6, nBPS, Z_{AB}|_H \neq 0, \text{large}} & \\ \underline{\mathcal{N} = 4} : & \text{SL}(2, \mathbb{R}) \times \frac{SO(6, 6)}{SO(1, 1) \times SO(5, 5)} & (4.24) \\ & \downarrow & \\ & \mathcal{O}_{\mathcal{N}=2, STU, nBPS, Z_H \neq 0, \text{large}} & \\ \underline{\mathcal{N} = 2} : & \frac{[SL(2, \mathbb{R})]^3}{[SO(1, 1)]^2} & \end{array}$$

For the numerators of cosets (4.19) holds, whereas for the stabilizers the following chain of embeddings holds:

$$E_{6(6)} \supsetneq SO(1, 1) \times SO(5, 5) \supsetneq [SO(1, 1)]^2 \times SO(4, 4). \quad (4.25)$$

In $\mathcal{N} = 2$ STU model, the manifest π -invariance of solution (4.22) corresponds to the fact that the central charge and matter charges are set on the very same footing. This leads to the statement that the corresponding four qubit state is four-way entangled [23]. In turn, this is consistent with the analysis of [31], stating that all families but $L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$ contain four-way entangled states.

On the other hand, the $\mathcal{N} = 2$ STU interpretation given above is confirmed by the fact that the non-translational part of the stabilizer of this nilpotent $SO_0(4, 4)$ -orbit, *i.e.* $[SO(1, 1; \mathbb{R})]^2$, coincides with the stabilizer of the rank-4 GHZ orbit (see [78], as well as Table VI of [46]) given by (4.24).

The fact that solution (4.22) is π -invariant corresponds to a Young tableaux which is invariant under $D = 3$ permutation symmetry. Consistently, the corresponding nilpotent $SO_0(4, 4)$ -orbit of real dimension 16 (related to L_{ab_3}) is unique [24], and it maps to the non-BPS $Z_H \neq 0$ “large” charge orbit of $\mathcal{N} = 2$, $D = 4$ STU model [78].

4.2.4 Conditions for Attractor Extremality

As discussed in Sect. VI of [23], all orbits treated in Sect. 4.2 have all four 4-qubit invariants vanishing. Namely, by using the notations of [23] (see *e.g.* App. VIII, as well as Refs., therein):

$$I_1 = I_2 = I_3 = I_4 = 0. \quad (4.26)$$

The (permutation-invariant) quadratic 4-qubit invariant I_1 has the physical interpretation of the extremality black hole parameter c^2 . Despite (4.26), the various 4-qubits states still can be characterized through their entanglement properties. Moreover,

the orbits of Subsect. 4.2.2 have $\mathcal{I}_4 > 0$, whereas the orbit of Subsect. 4.2.3 has $\mathcal{I}_4 < 0$, with \mathcal{I}_4 denoting the quartic (3-qubits [46]) $G_{4,STU}$ -invariant.

Conditions (4.2)-(4.3) are equivalent to (4.26): they both⁸ are conditions to have extremal ($c^2 = 0$) “large” ($\mathcal{I}_4 \neq 0$) STU $D = 4$ BHs, thus exhibiting an *Attractor Mechanism* ([82–86]; see Sect. VI of [23], as well).

As summarized in Subsect. 1.4 of [53] (which in turn recalls the treatment of [54]), the above conditions can be reformulated in a $D = 3$ language as follows: any static, spherically symmetric, asymptotically flat, extremal “large” black hole solution in $D = 4$ theories with symmetric scalar manifold is characterized by

$$[Q|_{\mathbf{R}}]^3 = 0, \quad (4.27)$$

where Q is the \mathfrak{g}_3 -valued $D = 3$ Noether charge, and \mathbf{R} is the relevant irrepr. of G_3 (for instance, the spinor $\mathbf{8}_s$ for $G_{3,STU}$). Among simple $D = 3$ U-duality groups related to symmetric scalar manifolds, the unique exception to nilpotency condition (4.27) is provided by $G_3 = E_{8(8)}$ (maximal supergravity, related to $J_3^{O_s}$), for which (4.27) gets replaced by

$$[Q|_{\mathbf{3875}}]^5 = 0. \quad (4.28)$$

It is also worth mentioning that in general, the condition (4.27) (or (4.28)) on Q must be supplemented by a condition on the conjugate $D = 3$ geodesic ($\mathfrak{g}_3 \ominus \mathfrak{h}_3^*$)-valued momentum (*cfr.* definition (1.15) of [53])

$$P \equiv \mathcal{V}Q\mathcal{V}^{-1}, \quad (4.29)$$

where \mathcal{V} is the $D = 3$ coset representative. Such a condition on P is discussed at the end of App. A.1 of [53], and it has recently been checked also in the $D = 3$ timelike reduction of the so-called $\mathcal{N} = 2$, $D = 4$ t^3 model in [81]⁹.

Moreover, it should be remarked that the treatment leading to nilpotency condition (4.27) (or (4.28)) is based on the assumption made in [54], that the extremal BHs under consideration can be obtained through a limit procedure from non-extremal black hole solutions. This observation will be reconsidered further below.

As treated in Sects. 5 and 6 of [54], discussed at the end of Subsect. 1.4 of [53], and stated at the end of Subsect. 2.4 of [53] itself, for all nilpotent G_3 -orbits satisfying the condition (4.27) (or (4.28)) *the non-translational part of the stabilizer coincides (at the horizon) with the stabilizer of the corresponding “large” orbit of the relevant $D = 4$ charge irrepr. of G_4 .* For $\mathcal{N} = 2$ “magic” octonionic ($J_3^{O_s}$) and $\mathcal{N} = 8$ (J_3^O) supergravity $D = 4$ theories¹⁰, this is expressed by Eqs. (2.100) and (2.101) of [53].

4.3 “Small” (*i.e.* Non-Attractor) Extremal $D = 4$ STU BHs

The theory of G_4 -orbits in the relevant (real, symplectic) charge representation space is known only for extremal¹¹ “large” and “small” $D = 4$ BHs in theories with symmetric scalar manifolds $\frac{G_4}{H_4}$, where $H_4 = mcs(G_4)$. In the supergravity theories related to

- $J_3^{O_s}$ (see App. A.2 of [53], as well as Eqs. (5.17) and (5.18) of [54]);
- J_3^H (see Eqs. (5.13)-(5.15) of [54]),

⁸As to our knowledge, the only possible counter-example to this statement might have been provided by the orbit \mathcal{O}'_{3K} of the $D = 3$ timelike-reduced S^3 model, studied in [81].

Indeed, orbit \mathcal{O}'_{3K} can be obtained from a “degeneration procedure” (outlined *e.g.* in Sect. 5 of [79], as well as at the very end of App. A.3 of [53]) of the orbits discussed in Sect. 4.2. As given by Table 2 of [81], it may also have $\mathcal{I}_4 = 0$ (besides $\mathcal{I}_4 > 0$).

However, just above the start of Subsect. 6.1.1 of [81], such an orbit is claimed to be *unphysical*, and thus to be disregarded.

⁹Namely, one obtains the coincidence of β -label and γ -label in physical solutions out of $SO_0(2, 2)$ -orbits \mathcal{O}_{3K} and \mathcal{O}'_{4K} ; see Subsect. 4.4 and Table 2 of [81].

¹⁰This matching can be explicitly checked for

1. $J_3^{O_s}$, by making use of the results reported in App. A.2 of [53] or equivalently, at the level of nilpotent $H_3^* \sim Spin^*(16)$ -strata, by considering Eqs. (5.17) and (5.18) of [54];
2. J_3^H , pertaining to the “twin” [87–89] theories $\mathcal{N} = 6$ and $\mathcal{N} = 2$ “magic” symmetric quaternionic. This case is considered in Subsect. 3.1 of [53], and the matching of orbit stabilizers can be checked, equivalently at the level of nilpotent $H_3^* \sim (Spin^*(12) \times SL(2, \mathbb{R}))$ -strata, from Eqs. (5.13)-(5.15) of [54].

¹¹Let us point out that in ([54] and) [53] “extremal” is used as synonym of “large with $c^2 = 0$ ”, whereas in [23] and in [81] “extremal” is used simply as synonym of “ $c^2 = 0$ ”, and we will adopt this latter use. In fact, note that in [81] various nilpotent $\tilde{K} \equiv H_{3,t^3}^* = SO_0(2, 2)$ -orbits correspond to BHs with $c^2 = 0$ and $\mathcal{I}_4 = 0$, thus to “small” extremal BHs.

the nilpotent G_3 -orbits (or, equivalently, their relevant Lagrangian submanifolds given by the corresponding nilpotent H_3^* -orbits/strata) related to extremal “small” $D = 4$ BHs have real dimension *smaller* than the ones related to extremal “large” $D = 4$ BHs, *i.e.* than the ones satisfying condition (4.27) (or (4.28)).

Furthermore, in the aforementioned cases the *mcs* of the non-translational part of the stabilizer of each of these nilpotent G_3 -orbits can be checked to match the *mcs* of the non-translational part of the stabilizer of the corresponding “small” G_4 -orbit, *i.e.* the stabilizer of the corresponding *moduli space* (if any) of $D = 4$ ADM mass. The corresponding $D = 4$ BHs are “small” and extremal, and therefore they all have $\mathcal{I}_4 = 0$ and $c^2 = 0$. This latter relation implies $I_1 = 0$, where I_1 is the quadratic 4-qubit invariant (see *e.g.* [23] and Refs. therein).

In the $\mathcal{N} = 2$ *STU* model, the situation is rather peculiar, because the groups involved are small, and they may also lack a non-trivial *mcs*. Actually, the three non-translational part of the stabilizers of rank-3 (*lightlike*), 2 (*critical*), 1 (*doubly-critical*) orbits of Table VI of [46] respectively are: \mathbb{I} , $(\text{SO}(2, 1))$ and $[\text{SO}(1, 1)]^2$, with *mcs* given by \mathbb{I} , $\text{SO}(2)$ and \mathbb{I} , respectively.

All this leads to the following statement: within the considered framework, the G_3 -nilpotent orbits with dimension *smaller* (corresponding to a nilpotency degree *lower*) than the one of the G_3 -nilpotent orbits satisfying condition (4.27) (or (4.28)) correspond to “small” extremal $D = 4$ BHs. As a consequence, the sets of G_3 -nilpotent orbits (grouped under $D = 3$ permutation symmetry) are in one-to-one correspondence with the classes of “small” charge orbits of G_4 .

Within the $\mathcal{N} = 2$ *STU* model, we are now going to work out in detail the correspondence among the rank-3, 2, 1 orbits of Table VI of [46] and the (classes of) nilpotent $\text{SO}_0(4, 4)$ -orbits of real dimension 16, 12 and 10, outlined in [24]. The *STU* model turns out to exhibit an high degree of “degeneration”: the BPS and non-BPS 3-charge orbits all are isomorphic, as well as all 2-charge orbits are. Furthermore, the $D = 3$ permutation properties of the related Young tableaux can be inferred from the action of cyclic permutations π on the representative solutions of the corresponding $G_{4,STU} = [\text{SL}(2, \mathbb{R})]^3$ -invariant constraints defining the “small” orbits of the $(2, 2, 2)$ irrepr. of $G_{4,STU}$.

4.3.1 A-W Classes $\Leftrightarrow L_{a_2 0_3 \oplus \bar{1}} : \text{Lightlike } \frac{1}{2}\text{-BPS and non-BPS Orbits}$

The 3-charge (*lightlike*) “small” orbit of the $(2, 2, 2)$ of $G_{4,STU}$ is given by (see Table VI of [46])

$$\mathcal{O}_{STU, \text{lightlike}} = \frac{[\text{SL}(2, \mathbb{R})]^3}{\mathbb{R}^2}, \quad \dim_{\mathbb{R}} = 7. \quad (4.30)$$

As given by the treatment below, such an orbit actually corresponds to three isomorphic orbits, which for a generic element of the Jordan symmetric sequence $\mathbb{R} \oplus \Gamma_{1, n_V - 2}$ with $n_V \geq 4$, are distinct [90].

The $[\text{SL}(2, \mathbb{R})]^3$ -invariant constraint which defines $\mathcal{O}_{STU, \text{lightlike}}$ is simply the vanishing of \mathcal{I}_4 :

$$\mathcal{I}_4 = \sum_i |z_i|^4 - 2 \sum_{i < j} |z_i|^2 |z_j|^2 + 4 \left(\prod_i z_i + \prod_i \bar{z}_i \right) = 0. \quad (4.31)$$

As a consequence of Eq. (4.9), the constraint (4.31) is manifestly π -invariant.

A set of representative solutions to the constraint (4.31) reads:

$$\begin{cases} |z_i| \equiv \mathcal{A}; \\ |z_{i+1}| = |z_{i+2}| = |z_{i+3}| \equiv \mathcal{B} \neq \mathcal{A}, \end{cases} \quad (4.32)$$

with $\mathcal{A}, \mathcal{B} \in \mathbb{R}_0^+$, and

$$\mathcal{A}^4 - 3\mathcal{B}^4 - 6\mathcal{A}^2\mathcal{B}^2 + 8\mathcal{A}\mathcal{B}^3 \cos \varphi = 0. \quad (4.33)$$

The z_i ’s are generally complex. The solutions (4.32)-(4.33) exhibit the maximal compact symmetry consistent with [73, 74]

$$\mathcal{O}_{\mathcal{N}=8, \frac{1}{8}\text{-BPS, small}} = \frac{E_{7(7)}}{F_{4(4)} \rtimes \mathbb{R}^{26}}, \quad (4.34)$$

namely $\text{SU}(2) \times \text{USp}(6) = \text{mcs}(F_{4(4)})$.

Another set of four representative solutions to the constraint (4.31) is given by

$$\begin{cases} z_i = 0; \\ |z_{i+1}|^4 + |z_{i+2}|^4 + |z_{i+3}|^4 - 2 \left(|z_{i+1}|^2 |z_{i+2}|^2 + |z_{i+1}|^2 |z_{i+3}|^2 + |z_{i+2}|^2 |z_{i+3}|^2 \right) = 0. \end{cases} \quad (4.35)$$

The z_i 's of solutions (4.35) are generally complex. The solutions (4.35) exhibit the generic compact symmetry (consistent with the structure of the skew-diagonalized $\mathcal{N} = 8, D = 4$ central charge matrix itself) $[\text{SU}(2)]^4$ (recall that $\text{USp}(2) = \text{SU}(2)$).

In both sets (4.32)-(4.33) and (4.35), the four solutions are related through the iterated action of π . This exactly corresponds to the $D = 3$ permutation properties of the Young tableaux of the $\text{SO}_0(4, 4)$ -nilpotent orbits of real dimension 16, in turn related to the four A - W classes belonging to the family $L_{a_2 0_3 \oplus \bar{1}}$ [24]: such Young tableaux are related through $D = 3$ permutation symmetry, and they reduce to a unique one, and thus to a unique $\text{SO}_0(4, 4)$ -nilpotent orbit, up to $D = 3$ permutations.

A direct comparison of (4.32) and (4.12) explains the analogous transformation properties under π , as well as the analogous structure of Young tableaux characterizing the set (4.32) of representative 3-charge solutions and the set (4.12) of attractor solutions with $\mathcal{I}_4 > 0$ (also recall that (4.32) does not admit $\mathcal{AB} = 0$). Indeed, the limit $\mathcal{B} = 0$ of Eq. (4.32) leads to Eq. (4.12), and the corresponding (maximal) manifest compact symmetry gets enhanced from $\text{SU}(2) \times \text{USp}(6)$ (pertaining to $\mathcal{O}_{\mathcal{N}=8, \frac{1}{8}-\text{BPS}, \text{small}}$) to $\text{SU}(2) \times \text{SU}(6)$ (pertaining to $\mathcal{O}_{\mathcal{N}=8, \frac{1}{8}-\text{BPS}, \text{large}}$).

It is here worth remarking that, despite they share the same supersymmetry properties in $\mathcal{N} = 8, D = 4$ supergravity, Eqs. (4.12) and (4.32) (or (4.35)) have different space-time properties, related to the *Attractor Mechanism* [82–86]:

- Eq. (4.12) has a space-time localization at the black hole event horizon, where the symmetry enhancement $[\text{SU}(2)]^4 \rightarrow \text{SU}(2) \times \text{SU}(6)$ due to *Attractor Mechanism* takes place. Furthermore, Eq. (4.12) is a solution to the $\text{SU}(8)$ -invariant Attractor Eqs., and it stabilizes the scalars in terms of the charges, fixing one point in the scalar manifold (for a given input of charges).
- Eq. (4.32) holds all along the scalar flow (for every value of the radial coordinate r), because, since the corresponding extremal black hole is “small”, there’s no event horizon at which the scalars should be stabilized. Furthermore, since Eq. (4.32) is a complete set of representative solutions to the G_4 -invariant constraint $\mathcal{I}_4 = 0$, it does not stabilize the scalars in terms of the charges, and thus it holds in all scalar manifold.

The difference in space-time localization and “scalar-manifold-localization” of Eqs. (4.12) and (4.32) is originated by the interplay between the *Attractor Mechanism* and the U-duality (*i.e.* G_4 -)invariance.

In the treatment below (refining the results of [24]), we show how the four Young tableaux associated to $L_{a_2 0_3 \oplus \bar{1}}$ can be related to one $\frac{1}{2}$ -BPS and three non-BPS 3-charge representative solutions of STU model, related through the iterated action of π . Such a $D = 4$ supersymmetry interpretation can be summarized by the following scheme [90]:

$$\begin{array}{ccc}
 \underline{\mathcal{N} = 8} : & \mathcal{O}_{\mathcal{N}=8, \frac{1}{8}-\text{BPS}, \text{small}} & \\
 & \downarrow & \searrow \\
 \underline{\mathcal{N} = 4} : & \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{C1}} \left[\frac{1}{4}-\text{BPS} \right] & \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{C2}} [n\text{BPS}] \\
 & \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6, 5)}{\text{SO}(4, 5) \times (\mathbb{R}^{4, 5} \times \mathbb{R})} & \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6, 5)}{\text{SO}(5, 4) \times (\mathbb{R}^{5, 4} \times \mathbb{R})} \\
 & \downarrow & \downarrow \\
 \underline{\mathcal{N} = 2} : & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C1}} \left[\frac{1}{2}-\text{BPS} \right] & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C2}} [n\text{BPS}], \\
 & \updownarrow * & \\
 & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C1}} [n\text{BPS}] &
 \end{array} \tag{4.36}$$

where “C1” and “C2” refer to the classification of [68, 76, 91], and it holds that

$$\mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C1}} \left[\frac{1}{2}-\text{BPS} \right] \sim \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C1}} [n\text{BPS}] \sim \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C2}} [n\text{BPS}] \tag{4.37}$$

$$\sim \mathcal{O}_{STU, \text{lightlike}} = \frac{[\text{SL}(2, \mathbb{R})]^3}{\mathbb{R}^2}, \tag{4.38}$$

consistent with Table VI of [46]. Consequently, Eqs. (4.36)-(4.38) correspond to the following chains of embeddings: for the numerators of cosets (4.19) holds, whereas for the stabilizers it holds $(\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2)$:

$$\begin{aligned}
 F_{4(4)} \times \mathbb{R}^{26} & \supsetneq \text{SO}(5, 4) \times (\mathbb{R}^{16} \times \mathbb{R}^{5, 4} \times \mathbb{R}) \\
 & \supsetneq \text{SO}(4, 4) \times (\mathbb{R}^{8_s} \times \mathbb{R}^{8_c} \times \mathbb{R}^{8_v} \times \mathbb{R}^2).
 \end{aligned} \tag{4.39}$$

In (4.39), the spinor \mathbb{R}^{16} and the *triality*-symmetric product $\mathbb{R}^{8_s} \times \mathbb{R}^{8_c} \times \mathbb{R}^{8_v}$ are progressively truncated out.

We now give a set of four independent representative solutions to lightlike constraint (4.31). They are particular, maximally-symmetric solutions of the type (4.35). We will explain their relation to the four Young tableaux of the four $\text{SO}_0(4, 4)$ -nilpotent orbits of real dimension 16, in turn related to the four A - W classes belonging to the family $L_{a_2 0_3 \oplus \bar{1}}$ [24].

As mentioned above, within the subsequent analysis, we will use the ($\mathcal{N} = 2$ STU analogue of the) $\mathcal{N} = 4$ normal frame adopted in [68], in which the relation to $\mathcal{N} = 8$, $D = 4$ supergravity (and the corresponding *quaternionality* properties) are more manifest.

1. The first prototypical representative solution reads

$$[1.i] : \begin{cases} z_1 = 0; \\ 2|z_3| = 2|z_4| = |z_2|. \end{cases} \quad (4.40)$$

This is a non-BPS solution: it is the $z_1 = 0$ limit of the representative solution of orbit $\mathcal{O}_{\mathcal{N}=2,STU,C1} [nBPS]$, given by the ($\mathcal{N} = 2$ STU analogue of the) “ $z_2 > z_1$ counterpart” of the solution treated at point C1) of Subsect. 4.4 of [68]. By applying triality transformation τ defined in (4.4), from [1.i] one can generate other two equivalent $\mathcal{N} = 2$ non-BPS solutions, belonging to $\mathcal{O}_{\mathcal{N}=2,STU,C1} [nBPS]$ itself, namely:

$$[1.ii] : \begin{cases} z_1 = 0; \\ 2|z_4| = 2|z_2| = |z_3|. \end{cases} \quad (4.41)$$

$$[1.iii] : \begin{cases} z_1 = 0; \\ 2|z_2| = 2|z_3| = |z_4|. \end{cases} \quad (4.42)$$

Solutions [1.i]-[1.iii] are τ -equivalent, because it holds that:

$$[1.i] \xrightarrow{\tau} [1.ii] \xrightarrow{\tau} [1.iii] \xrightarrow{\tau} [1.i]. \quad (4.43)$$

They all are $\mathcal{N} = 2$ non-BPS, belonging to the orbit $(\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2)$ [90]

$$\mathcal{O}_{\mathcal{N}=2,STU,C1} [nBPS] \sim \frac{[\text{SL}(2, \mathbb{R})]^3}{\mathbb{R}^2} = \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(2, n)}{\text{SO}(n-1) \ltimes (\mathbb{R}^{n-1} \times \mathbb{R})} \Big|_{n=2}. \quad (4.44)$$

According to (4.36), the supersymmetry reduction from maximal $D = 4$ supergravity reads

$$\mathcal{O}_{\mathcal{N}=8, \frac{1}{8}\text{-BPS}, \text{small}} \longrightarrow \mathcal{O}_{\mathcal{N}=4, n_V=6, C1} [\frac{1}{4}\text{-BPS}] \longrightarrow \mathcal{O}_{\mathcal{N}=2,STU,C1} [nBPS]. \quad (4.45)$$

The $\mathcal{N} = 4$ origin is also confirmed by the symmetry enhancement due to the fact that in solutions [1.i]-[1.iii] two $|z_i|$'s are equal:

$$[\text{SU}(2)]^4 \longrightarrow [\text{SU}(2)]^2 \times \text{USp}(4) \sim \text{SO}(4) \times \text{SO}(5) = mcs(\text{SO}(4, 5)). \quad (4.46)$$

Note that

$$|z_a| - |z_1| > 0, \quad \forall a = 1, 2, 3, \quad (4.47)$$

all along $\mathcal{O}_{\mathcal{N}=2,STU,C1} [nBPS]$. In terms of invariants of the STU model, constraint (4.47) can be written as (see notation used in [91], and Refs. therein)

$$i_a - i_1 > 0, \quad \forall \mathbf{a} = s, t, u. \quad (4.48)$$

Solutions [1.i]-[1.iii] correspond to the three solutions with maximal symmetry (in which two $|z_i|$'s are equal out of three) of (4.35) with $i = 1$, reading

$$\begin{cases} z_1 = 0; \\ |z_2|^4 + |z_3|^4 + |z_4|^4 - 2(|z_2|^2|z_3|^2 + |z_2|^2|z_4|^2 + |z_3|^2|z_4|^2) = 0. \end{cases} \quad (4.49)$$

[1.i]-[1.iii] are maximally symmetric solutions respectively to the three ways in which the latter quartic constraint can be rewritten:

$$\begin{aligned} & |z_2|^4 + |z_3|^4 + |z_4|^4 - 2(|z_2|^2|z_3|^2 + |z_2|^2|z_4|^2 + |z_3|^2|z_4|^2) \\ &= \begin{cases} [1.i] : & (|z_2|^2 - |z_3|^2 - |z_4|^2)^2 = 4|z_3|^2|z_4|^2; \\ & or \\ [1.ii] : & (-|z_2|^2 + |z_3|^2 - |z_4|^2)^2 = 4|z_2|^2|z_4|^2; \\ & or \\ [1.iii] : & (-|z_2|^2 - |z_3|^2 + |z_4|^2)^2 = 4|z_2|^2|z_3|^2. \end{cases} \end{aligned} \quad (4.50)$$

2. The second prototypical representative solution reads

$$[\mathbf{2.i}] : \begin{cases} z_2 = 0; \\ 2|z_3| = 2|z_4| = |z_1|. \end{cases} \quad (4.51)$$

This is a $(\frac{1}{2})$ -BPS solution: it is the $z_2 = 0$ limit of the representative solution of orbit $\mathcal{O}_{\mathcal{N}=2,STU,C1}$ [BPS], given by the ($\mathcal{N} = 2$ STU analogue of the) of the solution treated at point C1) of Subsect. 4.4 of [68]. By applying triality transformation τ , from [2.i] one can generate other two equivalent $\mathcal{N} = 2$ BPS solutions, belonging to $\mathcal{O}_{\mathcal{N}=2,STU,C1}$ [BPS] itself, namely:

$$[2.\text{ii}] \quad : \quad \begin{cases} z_3 = 0; \\ 2|z_4| = 2|z_2| = |z_1|. \end{cases} \quad (4.52)$$

$$[\mathbf{2.i}] \quad : \quad \left\{ \begin{array}{l} z_4 = 0; \\ 2|z_2| = 2|z_3| = |z_1|. \end{array} \right. \quad (4.53)$$

Solutions [2.i]-[2.iii] are τ -equivalent, because it holds that:

$$[\mathbf{2.i}] \xrightarrow{\tau} [\mathbf{2.ii}] \xrightarrow{\tau} [\mathbf{2.iii}] \xrightarrow{\tau} [\mathbf{2.i}]. \quad (4.54)$$

They all are $\mathcal{N} = 2$ BPS, belonging to the orbit [90]

$$\mathcal{O}_{N=2,STU,C1 [BPS]} \sim \frac{[\mathrm{SL}(2, \mathbb{R})]^3}{\mathbb{R}^2} = \mathrm{SL}(2, \mathbb{R}) \times \frac{\mathrm{SO}(2, n)}{\mathrm{SO}(n-1) \ltimes (\mathbb{R}^{n-1} \times \mathbb{R})} \Big|_{n=2}. \quad (4.55)$$

According to (4.36), the supersymmetry reduction from maximal $D = 4$ supergravity reads

$$\mathcal{O}_{\mathcal{N}=8, \frac{1}{8}\text{-BPS, small}} \longrightarrow \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{C1} [\frac{1}{4}\text{-BPS}]} \longrightarrow \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{C1} [BPS]}. \quad (4.56)$$

The $\mathcal{N} = 4$ origin is also confirmed by the symmetry enhancement (4.46), due to the fact that in solutions [2.i]-[2.iii] two $|z_i|$'s are equal. Respectively, solutions [2.i], [2.ii] and [2.iii] are solutions with maximal symmetry (with $|z_3| = |z_4|$, $|z_2| = |z_4|$, and $|z_2| = |z_3|$) of the following re-writings of (4.35) for $i = 2$, $i = 3$ and $i = 4$:

$$[\mathbf{2.i}] \quad : \quad \begin{cases} z_2 = 0; \\ |z_1|^4 + |z_3|^4 + |z_4|^4 - 2 \left(|z_1|^2 |z_3|^2 + |z_1|^2 |z_4|^2 + |z_3|^2 |z_4|^2 \right) = 0; \\ \qquad\qquad\qquad \Updownarrow \\ -|z_1|^2 + |z_3|^2 + |z_4|^2 = -2|z_3||z_4|. \end{cases} \quad (4.57)$$

$$[\mathbf{2.i}] \quad : \quad \left\{ \begin{array}{l} z_3 = 0; \\ |z_1|^4 + |z_2|^4 + |z_4|^4 - 2 \left(|z_1|^2 |z_2|^2 + |z_1|^2 |z_4|^2 + |z_2|^2 |z_4|^2 \right) = 0; \\ \qquad \qquad \qquad \Updownarrow \\ -|z_1|^2 + |z_2|^2 + |z_4|^2 = -2|z_2||z_4|. \end{array} \right. \quad (4.58)$$

$$[\textbf{2.i}] : \begin{cases} z_4 = 0; \\ |z_1|^4 + |z_2|^4 + |z_3|^4 - 2\left(|z_1|^2|z_2|^2 + |z_1|^2|z_3|^2 + |z_2|^2|z_3|^2\right) = 0; \\ \quad \quad \quad \Updownarrow \\ -|z_1|^2 + |z_2|^2 + |z_3|^2 = -2|z_2||z_3|. \end{cases} \quad (4.59)$$

Notice that the two sets $[1.i]-[1.iii]$ and $[2.i]-[2.iii]$ of τ -equivalent solutions are related by the exchange $z_1 \longleftrightarrow z_2$. This is irrelevant in $\mathcal{N} = 4$ supersymmetry, because it amounts to exchanging the two skew-eigenvalues of the

The third prototypical representative solution reads

This is a non-BPS solution: it is the representative solution of orbit $\mathcal{O}_{\mathcal{N}=2,STU,C2} [n_{BPS}]$, namely the ($\mathcal{N} = 2$ *STU* analogue of the) representative solution discussed at point C2) of Subsect. 4.4 of [68]. In particular, **[3.i]** corresponds to the branch

$$[\mathbf{3.iii}] \quad : \quad \begin{cases} z_2 = 0; \\ 2|z_1| = 2|z_4| = |z_3|. \end{cases} \quad (4.63)$$

$$[\mathbf{3.i}] \quad : \quad \left\{ \begin{array}{l} z_4 = 0; \\ |z_1|^4 + |z_2|^4 + |z_3|^4 - 2 \left(|z_1|^2 |z_2|^2 + |z_1|^2 |z_3|^2 + |z_2|^2 |z_3|^2 \right) = 0; \\ \qquad \qquad \qquad \Updownarrow \\ |z_1|^2 - |z_2|^2 + |z_3|^2 = -2 |z_1| |z_3|. \end{array} \right. \quad (4.68)$$

$$[3.iii] : \begin{cases} z_2 = 0; \\ |z_1|^4 + |z_3|^4 + |z_4|^4 - 2 \left(|z_1|^2 |z_3|^2 + |z_1|^2 |z_4|^2 + |z_3|^2 |z_4|^2 \right) = 0; \\ \quad \quad \quad \updownarrow \\ |z_1|^2 - |z_3|^2 + |z_4|^2 = -2 |z_1| |z_4|. \end{cases} \quad (4.69)$$

4. The fourth prototypical representative solution reads

$$[4.i] : \begin{cases} z_4 = 0; \\ 2 |z_1| = 2 |z_2| = |z_3|. \end{cases} \quad (4.70)$$

This is another representative solution of orbit $\mathcal{O}_{\mathcal{N}=2, STU, C2} [nBPS]$, namely of the ($\mathcal{N} = 2$ STU analogue of the) representative solution discussed at point C2) of Subsect. 4.4 of [68]. In particular, [4.i] corresponds to the branch

$$|z_3| - |z_4| = i_t - i_u > 0 \quad (4.71)$$

of $\mathcal{O}_{\mathcal{N}=2, STU, C2} [nBPS]$ itself (see [91], also for notations of invariants). By applying triality transformation τ , from [4.i] one can generate other two equivalent $\mathcal{N} = 2$ non-BPS solutions, belonging to $\mathcal{O}_{\mathcal{N}=2, STU, C2} [nBPS]$ itself, namely:

$$[4.ii] : \begin{cases} z_2 = 0; \\ 2 |z_1| = 2 |z_3| = |z_4|. \end{cases} \quad (4.72)$$

$$[4.iii] : \begin{cases} z_3 = 0; \\ 2 |z_1| = 2 |z_4| = |z_2|. \end{cases} \quad (4.73)$$

Solutions [4.i]-[4.iii] are τ -equivalent, because it holds that:

$$[4.i] \xrightarrow{\tau} [4.ii] \xrightarrow{\tau} [4.iii] \xrightarrow{\tau} [4.i]. \quad (4.74)$$

They all are $\mathcal{N} = 2$ non-BPS, belonging to the orbit (4.65), with supersymmetry reduction from maximal $D = 4$ supergravity given by (4.66). The $\mathcal{N} = 4$ origin is also confirmed by the symmetry enhancement (4.46), due to the fact that in solutions [4.i]-[4.iii] two $|z_i|$'s are equal. Respectively, solutions [4.i], [4.ii] and [4.iii] are solutions with maximal symmetry (with $|z_1| = |z_2|$, $|z_1| = |z_3|$, and $|z_1| = |z_4|$) of the following re-writings of (4.35) for $i = 4$, $i = 2$ and $i = 3$:

$$[4.i] : \begin{cases} z_4 = 0; \\ |z_1|^4 + |z_2|^4 + |z_3|^4 - 2 \left(|z_1|^2 |z_2|^2 + |z_1|^2 |z_3|^2 + |z_2|^2 |z_3|^2 \right) = 0; \\ \quad \quad \quad \updownarrow \\ |z_1|^2 + |z_2|^2 - |z_3|^2 = -2 |z_1| |z_2|. \end{cases} \quad (4.75)$$

$$[4.ii] : \begin{cases} z_2 = 0; \\ |z_1|^4 + |z_3|^4 + |z_4|^4 - 2 \left(|z_1|^2 |z_3|^2 + |z_1|^2 |z_4|^2 + |z_3|^2 |z_4|^2 \right) = 0; \\ \quad \quad \quad \updownarrow \\ |z_1|^2 + |z_3|^2 - |z_4|^2 = -2 |z_1| |z_3|. \end{cases} \quad (4.76)$$

$$[4.iii] : \begin{cases} z_3 = 0; \\ |z_1|^4 + |z_2|^4 + |z_4|^4 - 2 \left(|z_1|^2 |z_2|^2 + |z_1|^2 |z_4|^2 + |z_2|^2 |z_4|^2 \right) = 0; \\ \quad \quad \quad \updownarrow \\ |z_1|^2 - |z_2|^2 + |z_4|^2 = -2 |z_1| |z_4|. \end{cases} \quad (4.77)$$

Notice that the two sets $[3.i]-[3.iii]$ and $[4.i]-[4.iii]$ of τ -equivalent solutions are related by the exchange $z_3 \longleftrightarrow z_4$, immaterial both in $\mathcal{N} = 4$ supergravity (due to $\mathcal{N} = 4$ supersymmetry) and in $\mathcal{N} = 2$ *STU* model (for *triality* symmetry).

By noting that the four independent solutions given (4.35) (*e.g.* for $i = 1, 2, 3, 4$) are related though the iterated action of *quaternality* permutation symmetry π defined in (4.5), and by recalling definitions (4.4) and (4.5), one can determine how the twelve representative solutions of type **1**, **2**, **3** and **4** treated above are related through (composition of) τ and π . One can present the resulting web of relations in four equivalent ways, corresponding to using each of the four sets **1**, **2**, **3** and **4** of three τ -equivalent representative solutions as the “pivot” (first column on the left) for the iterated application of (composition(s) of) τ and π :

I \equiv set **1** as “pivot” :

$$\begin{array}{ccccccccc}
 [1.i] & \xrightarrow{\pi} & [3.iii] & \xrightarrow{\pi} & [3.i] & \xrightarrow{\pi} & [2.iii] & \xrightarrow{\pi} & [1.i] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [1.ii] & \xrightarrow{\pi} & [4.ii] & \xrightarrow{\pi} & [2.ii] & \xrightarrow{\pi} & [3.ii] & \xrightarrow{\pi} & [1.ii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [1.iii] & \xrightarrow{\pi} & [2.i] & \xrightarrow{\pi} & [4.iii] & \xrightarrow{\pi} & [4.i] & \xrightarrow{\pi} & [1.iii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [1.i] & \xrightarrow{\pi} & [3.iii] & \xrightarrow{\pi} & [3.i] & \xrightarrow{\pi} & [2.iii] & \xrightarrow{\pi} & [1.i]
 \end{array} \tag{4.78}$$

II \equiv set **2** as “pivot” :

$$\begin{array}{ccccccccc}
 [2.i] & \xrightarrow{\pi} & [4.iii] & \xrightarrow{\pi} & [4.i] & \xrightarrow{\pi} & [1.iii] & \xrightarrow{\pi} & [2.i] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [2.ii] & \xrightarrow{\pi} & [3.ii] & \xrightarrow{\pi} & [1.ii] & \xrightarrow{\pi} & [4.ii] & \xrightarrow{\pi} & [2.ii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [2.iii] & \xrightarrow{\pi} & [1.i] & \xrightarrow{\pi} & [3.iii] & \xrightarrow{\pi} & [3.i] & \xrightarrow{\pi} & [2.iii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [2.i] & \xrightarrow{\pi} & [4.iii] & \xrightarrow{\pi} & [4.i] & \xrightarrow{\pi} & [1.iii] & \xrightarrow{\pi} & [2.i]
 \end{array} \tag{4.79}$$

III \equiv set **3** as “pivot” :

$$\begin{array}{ccccccccc}
 [3.i] & \xrightarrow{\pi} & [2.iii] & \xrightarrow{\pi} & [1.i] & \xrightarrow{\pi} & [3.iii] & \xrightarrow{\pi} & [3.i] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [3.ii] & \xrightarrow{\pi} & [1.ii] & \xrightarrow{\pi} & [4.ii] & \xrightarrow{\pi} & [2.ii] & \xrightarrow{\pi} & [3.ii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [3.iii] & \xrightarrow{\pi} & [3.i] & \xrightarrow{\pi} & [2.iii] & \xrightarrow{\pi} & [1.i] & \xrightarrow{\pi} & [3.iii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [3.i] & \xrightarrow{\pi} & [2.iii] & \xrightarrow{\pi} & [1.i] & \xrightarrow{\pi} & [3.iii] & \xrightarrow{\pi} & [3.i]
 \end{array} \tag{4.80}$$

IV \equiv set **4** as “pivot” :

$$\begin{array}{ccccccccc}
 [4.i] & \xrightarrow{\pi} & [1.iii] & \xrightarrow{\pi} & [2.i] & \xrightarrow{\pi} & [4.iii] & \xrightarrow{\pi} & [4.i] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [4.ii] & \xrightarrow{\pi} & [2.ii] & \xrightarrow{\pi} & [3.ii] & \xrightarrow{\pi} & [1.ii] & \xrightarrow{\pi} & [4.ii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [4.iii] & \xrightarrow{\pi} & [4.i] & \xrightarrow{\pi} & [1.iii] & \xrightarrow{\pi} & [2.i] & \xrightarrow{\pi} & [4.iii] \\
 \downarrow \tau & & & & & & & & \downarrow \tau \\
 [4.i] & \xrightarrow{\pi} & [1.iii] & \xrightarrow{\pi} & [2.i] & \xrightarrow{\pi} & [4.iii] & \xrightarrow{\pi} & [4.i]
 \end{array} \tag{4.81}$$

Notice that the first and the last rows and the first and the last columns of each array coincide, due to the idempotency properties of τ and π themselves. Moreover, the rows of different arrays are related by cyclical reshufflings. The analogous

π -patterns of the arrays **I** and **II**, as well as of arrays **III** and **IV**, can be explained through the different roles of z_1 within the corresponding sets (**1** and **2**, as well as **3** and **4**, respectively) of τ -equivalent representative solutions (see treatment above).

In particular, the second row of each array has the remarkable property of containing only representative solutions of the type **[A.ii]** (**A** = **1, 2, 3, 4**), related through iterated application of the *quaternality* permutation π . Thus, the four solutions of type **[A.ii]** (with **A** = **1, 2, 3** and **4**, one ($\frac{1}{2}$ -)BPS and three non-BPS in $\mathcal{N} = 2$ supersymmetry) can conveniently and consistently be taken in one-to-one correspondence with the four Young tableaux of the four $\text{SO}_0(4, 4)$ -nilpotent orbits of real dimension 16, in turn related to the four *A-W* classes belonging to the family $L_{a_2 0_3 \oplus \bar{1}}$.

4.3.2 *A-B-EPR* Classes $\Leftrightarrow L_{a_2 b_2}$: *Critical* $\frac{1}{2}$ -BPS and non-BPS Orbits

The 2-charge (*critical*) “small” orbit of the $(2, 2, 2)$ of $G_{4,STU}$ is given by (see Table VI of [46])

$$\mathcal{O}_{STU, \text{crit.}} = \frac{[\text{SL}(2, \mathbb{R})]^3}{\text{SO}(2, 1) \times \mathbb{R}}, \dim_{\mathbb{R}} = 5. \quad (4.82)$$

$\mathcal{O}_{STU, \text{crit.}}$ is defined by an $[\text{SL}(2, \mathbb{R})]^3$ -invariant set of constraints which involve first-order functional derivatives of \mathcal{I}_4 itself:

$$\frac{\partial \mathcal{I}_4}{\partial z_i} = 2z_i \bar{z}_i^2 - 2\bar{z}_i \sum_{j \neq i} |z_j|^2 + 4 \prod_{j \neq i} z_j = 0, \forall i. \quad (4.83)$$

Note that, for each fixed i , $\frac{\partial \mathcal{I}_4}{\partial z_i}$ is manifestly invariant under cyclic permutations of the index $j \neq i$. As a consequence, the whole set of four constraints (4.83) is manifestly π -invariant.

The four constraints (4.83) admit six representative solutions, namely:

$$\left\{ \begin{array}{ll} \textbf{I} : & z_1 = 0 = z_2, z_3 = z_4 \neq 0; \\ \textbf{II} : & z_1 = 0 = z_3, z_2 = z_4 \neq 0; \\ \textbf{III} : & z_1 = 0 = z_4, z_2 = z_3 \neq 0; \\ \textbf{IV} : & z_2 = 0 = z_3, z_1 = z_4 \neq 0; \\ \textbf{V} : & z_2 = 0 = z_4, z_1 = z_3 \neq 0; \\ \textbf{VI} : & z_3 = 0 = z_4, z_1 = z_2 \neq 0, \end{array} \right. \quad (4.84)$$

which can be split into two sets (each being separately π -invariant)

$$\begin{aligned} \textbf{I} \xrightarrow{\pi} \textbf{IV} \xrightarrow{\pi} \textbf{VI} \xrightarrow{\pi} \textbf{III} \xrightarrow{\pi} \textbf{I} : & \left\{ \begin{array}{l} z_i = z_{i+1} = 0; \\ z_{i+2} = z_{i+3} \neq 0; \end{array} \right. \\ \textbf{II} \xrightarrow{\pi} \textbf{V} \xrightarrow{\pi} \textbf{II} : & \left\{ \begin{array}{l} z_i = z_{i+2} = 0; \\ z_{i+1} = z_{i+3} \neq 0. \end{array} \right. \end{aligned} \quad (4.85)$$

The non-vanishing z_i 's given by solutions (4.84) are generally complex. The set (4.84) (or equivalently (4.85)) exhibits the maximal compact symmetry consistent with [73, 74]

$$\mathcal{O}_{\mathcal{N}=8, \frac{1}{4}\text{-BPS}} = \frac{E_{7(7)}}{(\text{SO}(6, 5) \ltimes \mathbb{R}^{32}) \times \mathbb{R}}, \quad (4.86)$$

namely $\text{SO}(6) \times \text{SO}(5) \sim \text{SU}(4) \times \text{USp}(4) = mcs(\text{SO}(6, 5))$.

The properties under the action of π indicated in (4.85) determine two sets of Young tableaux, with cardinality 4 and 2 respectively [24]. In each of these two sets, the Young tableaux are related through $D = 3$ permutation symmetry, and thus they can be identified up to $D = 3$ permutations. The six representative solutions (4.85) are related to the six *A-B-EPR* classes of 4-qubits entanglement, organized in the two sets of Young tableaux corresponding to two groupings $\text{SO}_0(4, 4)$ -nilpotent orbits of dimension 12 (identified up to π), both associated to the family $L_{a_2 b_2}$ of [30].

Out of the six Young tableaux associated to $L_{a_2 b_2}$, three correspond to $\frac{1}{2}$ -BPS 2-charge *STU* BHs, and three correspond to non-BPS 2-charges *STU* BHs, with an equal amount of supersymmetric and non-supersymmetric solutions in the two subsets

given in (4.85). Such a $D = 4$ supersymmetry interpretation can be summarized by the following scheme [90]

$$\begin{array}{ccccc}
\mathcal{N}=8: & & \mathcal{O}_{\mathcal{N}=8, \frac{1}{4}-BPS} & & \\
& \swarrow & \downarrow & \searrow & \\
\mathcal{N}=4: & \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{A1}} [\frac{1}{2}-BPS] & \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{B}} [\frac{1}{4}-BPS] & & \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{A2}} [nBPS] \\
& \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6,6)}{\text{SO}(5,6) \times \mathbb{R}} & \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6,6)}{\text{SO}(2,1) \times \text{SO}(4,4) \times \mathbb{R}} & & \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6,6)}{\text{SO}(6,5) \times \mathbb{R}} \\
& \downarrow & \downarrow & & \downarrow \\
& \mathbf{VI}: & \mathbf{IV, V}: & & \mathbf{I}: \\
\mathcal{N}=2: & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{A1}} [\frac{1}{2}-BPS] & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{B}} [\frac{1}{2}-BPS] & & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{A2}} [nBPS], \\
& & \updownarrow * & & \\
& & \mathbf{II, III}: & & \\
& & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{B}} [nBPS] & &
\end{array} \tag{4.87}$$

where “**A1**”, “**A2**” and “**B**” refer to the classification of [68, 76], and the Latin uppercase numbers denote the solutions given in (4.85). This is consistent with the results of [76] and [68]. It holds that

$$\mathcal{O}_{\mathcal{N}=2, STU, \mathbf{A1}} [\frac{1}{2}-BPS] \sim \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{B}} [\frac{1}{2}-BPS] \sim \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{B}} [nBPS] \tag{4.88}$$

$$\sim \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{A2}} [nBPS] \sim \mathcal{O}_{STU, crit.} = \frac{[\text{SL}(2, \mathbb{R})]^3}{\text{SO}(2, 1) \times \mathbb{R}}, \tag{4.89}$$

consistent with Table VI of [46]. Consequently, Eqs. (4.87)-(4.89) correspond to the following chains of maximal symmetric embeddings: for the numerators of cosets it holds (4.19), whereas for the stabilizers it holds:

$$(\text{SO}(6, 5) \rtimes \mathbb{R}^{32}) \times \mathbb{R} \supsetneq \left(\text{SO}(2, 1) \times \text{SO}(4, 4) \rtimes \left(\mathbb{R}^{(2, 8_s)} \times \mathbb{R}^{(2, 8_c)} \right) \right) \times \mathbb{R}, \tag{4.90}$$

where the double-spinors $\mathbb{R}^{(2, 8_s)}$ and $\mathbb{R}^{(2, 8_c)}$ are truncated out.

Note that the sets $\{\mathbf{IV}, \mathbf{V}, \mathbf{VI}\}$ and $\{\mathbf{I}, \mathbf{II}, \mathbf{III}\}$ are separately invariant under τ . Since τ always commutes with $D = 4$ supersymmetry, each set is characterised by a unique supersymmetry property, namely $\{\mathbf{IV}, \mathbf{V}, \mathbf{VI}\}$ is $\frac{1}{2}$ -BPS, whereas $\{\mathbf{I}, \mathbf{II}, \mathbf{III}\}$ is non-BPS. Thus, each of the two sets of Young tableaux corresponding to the solutions (4.85) has 50% supersymmetric and 50% non-supersymmetric contents; namely, the set $\{\mathbf{I}, \mathbf{III}, \mathbf{IV}, \mathbf{VI}\}$ contains two $\frac{1}{2}$ -BPS (\mathbf{IV} and \mathbf{VI}) and two non-BPS (\mathbf{I} and \mathbf{III}) 2-charge solutions, whereas the set $\{\mathbf{II}, \mathbf{V}\}$ contains one $\frac{1}{2}$ -BPS (\mathbf{V}) and one non-BPS (\mathbf{II}) 2-charge solution.

One can also check that the mcs of the non-translational part of the stabilizer of $\mathcal{O}_{STU, crit.}$ is (non-maximally) embedded into the mcs 's of the non-translational part of the stabilizers of the two $\text{SO}_0(4, 4)$ -nilpotent orbits of dimension 12, namely:

$$mcs((SO(2, 1)) = \text{SO}(2) \subsetneq \begin{cases} \text{SO}(3) \times \text{SO}(2) = mcs(\text{SO}(3, 2; \mathbb{R})); \\ \text{SO}(4) = mcs(Sp(4, \mathbb{R})). \end{cases} \tag{4.91}$$

4.3.3 A-B-C-D Class $\Leftrightarrow L_{abc_2}$: Doubly-Critical $\frac{1}{2}$ -BPS Orbit

The 1-charge (*doubly-critical*) “small” orbit of the $(2, 2, 2)$ of $G_{4, STU}$ is given by (see Table VI of [46])

$$\mathcal{O}_{STU, doubly-crit} = \frac{[\text{SL}(2, \mathbb{R})]^3}{[\text{SO}(1, 1)]^2 \rtimes \mathbb{R}^3}, \dim_{\mathbb{R}} = 4. \tag{4.92}$$

$\mathcal{O}_{STU, doubly-crit}$ is defined by an $[\text{SL}(2, \mathbb{R})]^3$ -invariant set of constraints which involve suitable projections of second-order functional derivatives of \mathcal{I}_4 itself (see *e.g.* [76, 91]). Such a set of constraints can be recast in the following form:

$$\begin{cases} |z_1|^2 = |z_2|^2 = |z_3|^2 = |z_4|^2; \\ z_i z_j - \overline{z_k} \overline{z_l} = 0, \forall i \neq j \neq k \neq l. \end{cases} \tag{4.93}$$

Note the similarity of the second of (4.93) with the Attractor Eqs. (4.8) themselves.

The constraints (4.93) are manifestly π -invariant. It should also be pointed out that out of the second set of constraints of (4.93), only the following Eqs. are independent:

$$\begin{cases} z_1 z_2 - \overline{z_3} \overline{z_4} = 0; \\ z_1 z_3 - \overline{z_2} \overline{z_4} = 0, \end{cases} \tag{4.94}$$

and all the others can be obtained through iterated action of π (and through complex conjugation).

Constraints (4.93) admit the following representative solutions, manifestly π -invariant:

$$\begin{cases} |z_i| = \eta e^{i\frac{\varphi}{4}}, \quad \eta \in \mathbb{R}_0^+, \quad \forall i, \\ \varphi = 2m\pi, \quad m \in \mathbb{Z}. \end{cases} \quad (4.95)$$

Notice the similarity of “small” 1-charge (*doubly-critical*) $\frac{1}{2}$ -BPS (both in $\mathcal{N} = 2$ and $\mathcal{N} = 8$) solution (4.95) with the “large” (and thus attractor) (non-BPS $Z_H \neq 0$ in $\mathcal{N} = 2$ and non-BPS in $\mathcal{N} = 8$) solution (4.22).

The solution (4.95) exhibits the maximal compact symmetry consistent with [73, 74]

$$\mathcal{O}_{\mathcal{N}=8, \frac{1}{2}\text{-BPS}} = \frac{E_{7(7)}}{E_{6(6)} \rtimes \mathbb{R}^{27}}, \quad (4.96)$$

namely $\text{USp}(8) = mcs(E_{6(6)})$.

The manifestly π -invariant solution (4.95) corresponds to a unique Young tableaux, manifestly invariant under $D = 3$ permutation symmetry, and related to the totally separable A - B - C - D class of 4-qubits entanglement. This in turn determines a unique $\text{SO}_0(4, 4)$ -nilpotent orbit, namely the *minimal* one of real dimension 10, corresponding to the family L_{abc_2} of 4-qubits entanglement states [24].

It generally holds that in $D = 4$ the 1-charge orbit is always unique and maximally supersymmetric (namely, $\frac{1}{2}$ -BPS): it corresponds to the *minimal* nilpotent G_3 -orbit.

Correspondingly, there exists a unique “small” 1-charge (*doubly-critical*) $\frac{1}{2}$ -BPS orbit in $\mathcal{N} = 2$, $D = 4$ STU model, given by (4.92). Such a $D = 4$ supersymmetry interpretation can be summarized as follows [90] (subscript denote $\text{SO}(1, 1)$ -weights):

$$\begin{array}{ccc} \underline{\mathcal{N} = 8} : & \mathcal{O}_{\mathcal{N}=8, \frac{1}{2}\text{-BPS}} & \\ & \downarrow & \\ & \mathcal{O}_{\mathcal{N}=4, n_V=6, \mathbf{A3}[\frac{1}{2}\text{-BPS}]} & \\ \underline{\mathcal{N} = 4} : & \text{SL}(2, \mathbb{R}) \times \frac{\text{SO}(6, 6)}{[\text{SO}(1, 1) \times \text{SO}(5, 5)] \rtimes (\mathbb{R}^{5, 5-2} \times \mathbb{R}^{1+4})} & \\ & \downarrow & \\ \underline{\mathcal{N} = 2} : & \mathcal{O}_{\mathcal{N}=2, STU, \mathbf{A3}[\frac{1}{2}\text{-BPS}]} & \end{array} \quad (4.97)$$

where (recall Eq. (4.92))

$$\mathcal{O}_{\mathcal{N}=2, STU, \mathbf{A3}[\frac{1}{2}\text{-BPS}]} \sim \mathcal{O}_{STU, doubly\text{-}crit} = \frac{[\text{SL}(2, \mathbb{R})]^3}{[\text{SO}(1, 1)]^2 \rtimes \mathbb{R}^3}. \quad (4.98)$$

Consequently, Eqs. (4.97)-(4.98) correspond to the following chains of maximal symmetric embeddings: for the numerators it holds (4.19), whereas for the stabilizers it holds $(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^3)$:

$$\begin{aligned} & E_{6(6)} \rtimes \mathbb{R}^{27} \\ \supseteq & \text{SO}(5, 5) \times \text{SO}(1, 1) \rtimes (\mathbb{R}^{5, 5-2} \times \mathbb{R}^{16+1} \times \mathbb{R}^{1+4}) \\ \supseteq & \text{SO}(4, 4) \times \text{SO}(1, 1) \times \text{SO}(1, 1) \\ & \rtimes (\mathbb{R}^{8_{v, -2, 0}} \times \mathbb{R}^{8_{c, +1, +1}} \times \mathbb{R}^{8_{s, +1, -1}} \times \mathbb{R}^{1-2, +2} \times \mathbb{R}^{1-2, -2} \times \mathbb{R}^{1+4, 0}) \end{aligned} \quad (4.99)$$

where the subscript denote $\text{SO}(1, 1)$ -weights, and the translational factors \mathbb{R}^{16+1} and $\mathbb{R}^{8_{v, -2, 0}} \times \mathbb{R}^{8_{c, +1, +1}} \times \mathbb{R}^{8_{s, +1, -1}}$ are progressively truncated out.

Finally, it is worth remarking that a comparison of (4.95) and (4.22) explains the π -invariance characterizing both the 1-charge solution (4.95) and the attractor solution (4.22) with $\mathcal{I}_4 < 0$.

4.4 “Extremal” $D = 4$ STU BHs

It should be remarked that, consistent with the assumption made in [54] and [53], the extremality characterizing the $D = 4$ “large” and “small” BHs associated to nilpotent $\text{SO}_0(4, 4)$ -orbits of real dimension 10, 12, 16 and 18 (treated in Sects. 4.2 and 4.3) is an extremality *which can be obtained through a limit process* from a non-extremal BH solution.

However, there exist extremal $D = 4$ BHs which cannot be seen as the “extremal limit” of non-extremal BH solutions. As done in [24], we dub them “extremal” BHs (i.e. with the quotation marks). An example of this type of BHs is provided by the

nilpotent $G_{3,t^3} (= G_{2(2)})$ -orbit \mathcal{O}_5 . As given by Table 1 and Fig. 2 (Hasse diagram of $G_{2(2)}$, with partial ordering relations) of [81], this orbit is the one with highest degree (namely, 7; *cfr.* Eq. (4.33) of [81]) of nilpotency, and it is therein claimed not to be given by the extremal limit of a non-extremal BH solution. Through the (inverse of the) embedding procedure (*cfr.* Eq. (A.41) of [53])

$$\begin{array}{c} G_{2(2)} \\ \subsetneq \\ G_{3,T^3} \end{array} \subsetneq \begin{array}{c} \text{SO}_0(4,3) \\ \subsetneq \\ G_{3,ST^2} \end{array} \subsetneq \begin{array}{c} \text{SO}_0(4,4) \\ \subsetneq \\ G_{3,STU} \end{array}, \quad (4.100)$$

which is discussed at the end of App. A.3 of [53], as well as in Sect. 5 of [79], the $G_{2(2)}$ -orbit \mathcal{O}_5 determines all nilpotent $\text{SO}_0(4,4)$ -orbits of real dimension larger than 18, namely 20, 22 and 24, as resulting from Hasse diagram of $\text{SO}_0(4,4)$, *e.g.* given by Fig. 1 of [24].

Since all nilpotent G_3 -orbits are characterised by all the four 4-qubit invariants vanishing (as resulting from page 14, Table 3 and Table 6 of [31]), the statement is that in general the $\text{SO}_0(4,4)$ -nilpotent orbits of dimension 20, 22 and 24 (respectively associated to the families L_{a_4} , $L_{0_{5\oplus\bar{3}}}$ and $L_{0_{7\oplus\bar{1}}}$) correspond to “extremal” (with $I_1 = 0$, “small” *and/or* “large”) $D = 4$ BHs, which are *not* the limit of non-extremal BH solutions.

The $\text{SO}_0(4,4)$ -nilpotent orbits corresponding to the families L_{ab_3} , L_{a_4} , $L_{0_{5\oplus\bar{3}}}$ and $L_{0_{7\oplus\bar{1}}}$ generally contain 4-way entangled states. However, since we are considering $\text{SO}_0(4,4)$ -orbits which are nilpotent, the corresponding parameters (*if any*) are all set to zero (consistent with the claim at page 14 of [31]).

We leave the study of such “extremal” BHs for further future investigation.

5 Four-Way Entanglement of Eight Qubits in $\mathcal{N} = 8$ Supergravity...

Having now seen in some detail how the BHs of the STU model are intricately related to the entanglement of three and four qubits, it is natural to ask whether this intriguing correspondence can be extended to other supergravity theories. Given that the STU model may be embedded in the $\mathcal{N} = 8$ theory this is a natural and interesting case to consider, especially given its $E_{7(7)}$ U-duality group, which is rather exotic from the perspective of quantum information theory. Indeed, the BHs of $\mathcal{N} = 8$ supergravity were related to qubits in [12, 13]. Since the BHs transform linearly under the U-duality group $E_{7(7)}$ they cannot simply correspond to the arbitrary entanglement of more qubits. Indeed, they are related to a very special tripartite entanglement of seven qubits as described by the Fano plane [12].

The maximally supersymmetric $D = 4, \mathcal{N} = 8$ supergravity [92] is based on the degree-3 Jordan algebra $J_3^{O_s}$ of 3×3 Hermitian matrices over the split form of the octonions O_s [93, 94]. It contains 70 scalar fields parametrising the coset (3.21), where $E_{7(7)}$ is the U-duality group and $\text{SU}(8)$ its maximal compact subgroup. There are also 28 gauge potentials, which, together with their 28 magnetic duals, transform linearly as the **56** of $E_{7(7)}$. The stationary BH solutions carry these charges and the extremal solutions have a Bekenstein-Hawking entropy given by

$$S = \pi \sqrt{|\mathcal{I}_4|}, \quad (5.1)$$

where \mathcal{I}_4 is the unique Cartan-Cremmer-Julia quartic invariant of $E_{7(7)}$ [92, 95] built from the 56 electromagnetic charges [66].

The crucial observation relating the black holes to the tripartite entanglement of seven qubits is that E_7 contains seven copies of the single qubit SLOCC group $\text{SL}(2)$ and that the **56** decomposes in a very particular way. Under

$$E_{7(7)} \supset \text{SL}(2)_A \times \text{SL}(2)_B \times \text{SL}(2)_C \times \text{SL}(2)_D \times \text{SL}(2)_E \times \text{SL}(2)_F \times \text{SL}(2)_G \quad (5.2)$$

the **56** decomposes as

$$\begin{aligned} \mathbf{56} \rightarrow & (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}) \\ & + (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}) \\ & + (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}). \end{aligned} \quad (5.3)$$

Note, each term in the above decomposition transforms as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ under three of the $\text{SL}(2)$ factors and as singlets under the remaining four, but taken together they transform as the **56** of $E_{7(7)}$. This translates into seven intertwined copies of the

3-qubit Hilbert space:

$$\begin{aligned}
|\Psi\rangle_{56} = & a_{ABD}|ABD\rangle \\
& + b_{BCE}|BCE\rangle \\
& + c_{CDF}|CDF\rangle \\
& + d_{DEG}|DEG\rangle \\
& + e_{EFA}|EFA\rangle \\
& + f_{FGB}|FGB\rangle \\
& + g_{GAC}|GAC\rangle.
\end{aligned} \tag{5.4}$$

This state has a very distinctive structure:

1. Two distinct qubits appear together in one and only one tripartite entanglement.
2. Any two tripartite entanglements have at least one qubit in common.
3. Every qubit belongs to three distinct tripartite entanglements.

On replacing the words qubit and tripartite entanglement with the words point and line, respectively, it becomes apparent that the state describes the projective plane of order 2. This is known as the Fano plane, which is depicted in Figure 1. The Fano plane

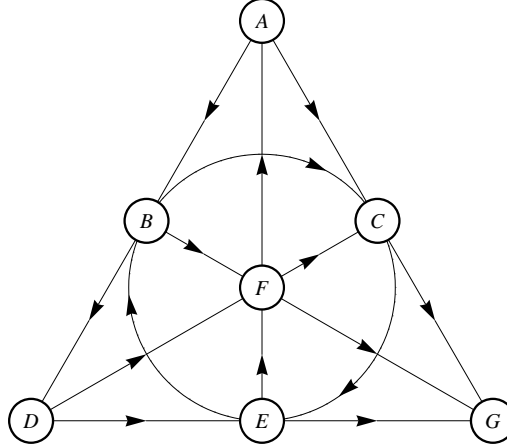


Figure 1: The Fano plane is a projective plane with seven points and seven lines (the circle counts as a line). We may associate it to the state (5.4) by interpreting the points as the seven qubits $A-G$ and the lines as the seven tripartite entanglements.

is also the multiplication table of the imaginary octonions. This special state has revealed a number of interesting connections relating exceptional groups, octonions and special finite geometries to quantum information theory and, in particular, three qubits [18–20, 96].

Repeating the analysis of the STU model for the maximally supersymmetric theory leads to another exotic qubit configuration: the four-way entanglement of eight qubits. A time-like reduction of the $\mathcal{N} = 8$ theory yields the scalar manifold given in (3.22), where $E_{8(8)}$ is the $D = 3$ U-duality group and $SO^*(16)$ is a non-compact form of its maximal compact subgroup. The stationary BH solutions are given by the geodesics in (3.22) that are in turn parametrised by $\mathfrak{e}_{8(8)}$ valued Noether charges. In particular, the extremal solutions correspond to the nilpotent orbits of $E_{8(8)}$ acting on $\mathfrak{e}_{8(8)}$. The Cartan decomposition

$$\mathfrak{e}_{8(8)} = \mathfrak{so}^*(16) \oplus \mathbf{128}, \tag{5.5}$$

where $\mathbf{128}$ is the spinor of $SO^*(16)$, implies, by the Kostant-Sekiguchi theorem, that the orbits the nilpotent orbits of $E_{8(8)}$ acting on $\mathfrak{e}_{8(8)}$ are in one-to-one correspondence to the nilpotent orbits of $SO(16, \mathbb{C})$ acting on $\mathbf{128}$ [52–54]. The 128 independent components are given by the $28 + 28$ electromagnetic charges, the NUT charge, the mass and 70 scalars of the $\mathcal{N} = 8$ theory.

The qubit interpretation is obtained by decomposing the adjoint (fundamental) of $E_{8(8)}$ with respect to $[\mathrm{SL}(2)]^8$ [19]. This can be interpreted as the time-like reduction of the tripartite entanglement of seven qubits, the eighth $\mathrm{SL}(2)$ being the Ehlers group. Explicitly,

$$E_8 \supset \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_C \times \mathrm{SL}(2)_D \times \mathrm{SL}(2)_E \times \mathrm{SL}(2)_F \times \mathrm{SL}(2)_G \times \mathrm{SL}(2)_H, \tag{5.6}$$

under which

$$\begin{aligned}
248 \rightarrow & (3, 1, 1, 1, 1, 1, 1, 1) + (2, 2, 2, 1, 2, 1, 1, 1) + (1, 1, 1, 2, 1, 2, 2, 2) \\
& + (1, 3, 1, 1, 1, 1, 1, 1) + (2, 1, 2, 2, 1, 2, 1, 1) + (1, 2, 1, 1, 2, 1, 2, 2) \\
& + (1, 1, 3, 1, 1, 1, 1, 1) + (2, 1, 1, 2, 2, 1, 2, 1) + (1, 2, 2, 1, 1, 2, 1, 2) \\
& + (1, 1, 1, 3, 1, 1, 1, 1) + (2, 1, 1, 1, 2, 2, 1, 2) + (1, 2, 2, 2, 1, 1, 2, 1) \\
& + (1, 1, 1, 1, 3, 1, 1, 1) + (2, 2, 1, 1, 1, 2, 2, 1) + (1, 1, 2, 2, 2, 1, 1, 2) \\
& + (1, 1, 1, 1, 1, 3, 1, 1) + (2, 1, 2, 1, 1, 1, 2, 2) + (1, 2, 1, 2, 2, 2, 1, 1) \\
& + (1, 1, 1, 1, 1, 1, 3, 1) + (2, 2, 1, 2, 1, 1, 1, 2) + (1, 1, 2, 1, 2, 2, 2, 1) \\
& + (1, 1, 1, 1, 1, 1, 1, 3),
\end{aligned} \tag{5.7}$$

and

$$\mathfrak{e}_{8(8)} \cong [\mathfrak{sl}(2, \mathbb{R})]^8 \oplus \mathfrak{p} \tag{5.8}$$

where

$$\begin{aligned}
\mathfrak{p} = & (2, 2, 2, 1, 2, 1, 1, 1) + (1, 1, 1, 2, 1, 2, 2, 2) \\
& + (2, 1, 2, 2, 1, 2, 1, 1) + (1, 2, 1, 1, 2, 1, 2, 2) \\
& + (2, 1, 1, 2, 2, 1, 2, 1) + (1, 2, 2, 1, 1, 2, 1, 2) \\
& + (2, 1, 1, 1, 2, 2, 1, 2) + (1, 2, 2, 2, 1, 1, 2, 1) \\
& + (2, 2, 1, 1, 1, 2, 2, 1) + (1, 1, 2, 2, 2, 1, 1, 2) \\
& + (2, 1, 2, 1, 1, 1, 2, 2) + (1, 2, 1, 2, 2, 2, 1, 1) \\
& + (2, 2, 1, 2, 1, 1, 1, 2) + (1, 1, 2, 1, 2, 2, 2, 1),
\end{aligned} \tag{5.9}$$

which admits an interpretation as the four-way entanglement of eight qubits,

$$\begin{aligned}
|\Psi\rangle_{224} = & a_{HABD}|HAB \bullet D \bullet \bullet \bullet\rangle + \tilde{a}_{CEFG}|\bullet \bullet \bullet C \bullet EFG\rangle \\
& + b_{HBCE}|H \bullet BC \bullet E \bullet \bullet\rangle + \tilde{b}_{DFGA}|\bullet A \bullet \bullet D \bullet FG\rangle \\
& + c_{HCDF}|H \bullet \bullet CD \bullet F \bullet\rangle + \tilde{c}_{EGAB}|\bullet AB \bullet \bullet E \bullet G\rangle \\
& + d_{HDEG}|H \bullet \bullet \bullet DE \bullet G\rangle + \tilde{d}_{FABC}|\bullet ABC \bullet \bullet F \bullet\rangle \\
& + e_{HEFA}|HA \bullet \bullet \bullet EF \bullet\rangle + \tilde{e}_{GBCD}|\bullet \bullet BCD \bullet \bullet \bullet\rangle \\
& + f_{HFG B}|H \bullet B \bullet \bullet \bullet FG\rangle + \tilde{f}_{ACDE}|\bullet A \bullet CDE \bullet \bullet\rangle \\
& + g_{HGAC}|HA \bullet C \bullet \bullet \bullet G\rangle + \tilde{g}_{BDEF}|\bullet \bullet B \bullet DEF \bullet\rangle.
\end{aligned} \tag{5.10}$$

Half the states are given by the quadrangles of the Fano plane and the other half by the quadrangles of the dual Fano plane. See also [97], where this configuration was related to a *doubled Fano plane*. While we may assign $|\Psi\rangle_{224}$ to the coset¹² $E_{8(8)}/[\mathrm{SL}(2, \mathbb{R})]^8$, unlike the *STU* example, the Kostant-Sekiguchi theorem does *not* apply. Indeed, $E_{8(8)}/[\mathrm{SL}(2, \mathbb{R})]^8$ is not a symmetric space, as it can be verified *e.g.* by considering the non-zero commutation relations of, for example, two elements in $(2, 2, 2, 1, 2, 1, 1, 1)$ and $(2, 1, 2, 2, 1, 2, 1, 1)$. This is in fact as one would anticipate, since in four dimensions the tripartite entanglement of seven qubits forms a representation of the full U-duality group, not just its $[\mathrm{SL}(2)]^7$ subgroup. Consequently, performing the time-like reduction, it is actually the nilpotent orbits of $\mathrm{SO}(16, \mathbb{C})$ acting on its spinorial representation that are of relevance.

6 ...and in $\mathcal{N} = 2$ Exceptional Supergravity

Another interesting case to consider is the $\mathcal{N} = 2$ exceptional supergravity, namely the magic model based on the degree-3 Euclidean Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over the division algebra of octonions \mathbb{O} [93, 94]. This is the unique magic model which cannot be obtained as consistent truncation of the maximal theory treated in previous Section. The extension of the connection between BHs and QIT to the case of magic supergravity was firstly suggested by Levey in [13] (based on work of [97] and [98]), and it has been considered by one of the present authors and Ferrara in [14]. The U-duality group of the $D = 4$ exceptional theory is another non-compact, real form of E_7 , namely $E_{7(-25)}$, which, as its maximal counterpart $E_{7(7)}$, is rather exotic from a QIT perspective, as well. Once again, since the BHs transform linearly under the

¹²Note that $E_{8(8)}$ contains (by a chain of maximal symmetric embeddings) $[\mathrm{SL}(2, \mathbb{R})]^8$, $[\mathrm{SL}(2, \mathbb{R})]^4 \times [\mathrm{SU}(2)]^4$, and $[\mathrm{SU}(2)]^8$. These correspond to different completions of the 8-dimensional Cartan subalgebra of $E_{8(8)}$. However, if one constrains the first inclusion of the chain to be $E_{8(8)} \supset E_{7(7)} \times \mathrm{SL}(2, \mathbb{R})$, then the embedding of $[\mathrm{SU}(2)]^8$ is excluded.

U-duality group $E_{7(-25)}$, they are not expected to be related to an arbitrary entanglement of more qubits. As it will become evident from treatment below, after a timelike reduction to $D = 3$ they result to be related to a curious combination of local unitary and special linear factor groups within an entanglement of eight qubits with the same structure of the one treated in previous Section.

The exceptional magic $D = 4$, $\mathcal{N} = 2$ supergravity [93, 94] has 27 complex scalar fields (one for each Abelian vector multiplet), parametrising the rank-3 special Kähler symmetric coset

$$\frac{E_{7(-25)}}{E_{6(-78)} \times \mathrm{U}(1)}, \quad (6.1)$$

where $E_{7(-25)}$ is the U-duality group and $E_{6(-78)} \times \mathrm{U}(1)$ its maximal compact subgroup. There $28 = 1$ (graviphoton) + 27 gauge potentials, together with their 28 magnetic duals, transform linearly as the **56** of $E_{7(-25)}$. The stationary BH solutions carry these charges and the extremal solutions have a Bekestein-Hawking entropy given by the same formula (5.1) of the $\mathcal{N} = 8$ case, where \mathcal{I}_4 is now the unique quartic invariant of $E_{7(-25)}$ built of the 56 electromagnetic charges.

When considering the groups in the complex field, *mutatis mutandis* the story goes as in the maximal theory treated in previous Section, but the interpretation in terms of timelike reduction down to $D = 3$ is different. Indeed, by performing such a reduction, the $\mathcal{N} = 2$ exceptional theory yields the scalar manifold to become the rank-4 para-quaternionic, pseudo-Riemannian symmetric coset

$$\frac{E_{8(-24)}}{E_{7(-25)} \times \mathrm{SL}(2, \mathbb{R})}, \quad (6.2)$$

which is obtained from (6.1) through the so-called c^* -map ([58], [52], and Refs. therein). $E_{8(-24)}$ is the $D = 3$ U-duality group and $E_{7(-25)} \times \mathrm{SL}(2, \mathbb{R})$ is a non-compact form of its maximal compact subgroup, the factor $\mathrm{SL}(2, \mathbb{R})$ being the Ehlers group¹³ The stationary BH solutions are given by the geodesics in (6.2) that are in turn parametrised by $\mathfrak{e}_{8(-24)}$ valued Noether charges. In particular, the extremal solutions correspond to the nilpotent orbits of $E_{8(-24)}$ acting on $\mathfrak{e}_{8(-24)}$. The Cartan decomposition

$$\mathfrak{e}_{8(-24)} = (\mathfrak{e}_{7(-25)} + \mathfrak{sl}(2, \mathbb{R})) \oplus (\mathbf{56}, \mathbf{2}), \quad (6.3)$$

implies, by the Kostant-Sekiguchi theorem, that the nilpotent orbits of $E_{8(-24)}$ acting on $\mathfrak{e}_{8(-24)}$ are in one-to-one correspondence to the nilpotent orbits of $E_7(\mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ acting on $(\mathbf{56}, \mathbf{2})$. In the real field, these latter has 112 independent components, given by the $28 + 28$ electromagnetic charges, the NUT charge, the mass and 54 (27 complex) scalar degrees of freedom of the $\mathcal{N} = 2$, $D = 4$ exceptional theory.

The qubit interpretation is obtained by decomposing the adjoint (fundamental) of $E_{8(-24)}$ with respect to $[\mathrm{SL}(2, \mathbb{R})]^4 \times [\mathrm{SU}(2)]^4$. Indeed, differently from $E_{8(8)}$, $E_{8(-24)}$ contains (by a chain of maximal symmetric embeddings) $[\mathrm{SL}(2, \mathbb{R})]^4 \times [\mathrm{SU}(2)]^4$ and $[\mathrm{SU}(2)]^8$ (corresponding to different completions of the 8-dimensional Cartan subalgebra of $E_{8(-24)}$), but not $[\mathrm{SL}(2, \mathbb{R})]^8$. If one further constrains the first inclusion of the chain to be the relevant one for c^* -map $E_{8(-24)} \supset E_{7(-25)} \times \mathrm{SL}(2, \mathbb{R})$, then also the embedding of $[\mathrm{SU}(2)]^8$ is excluded. Explicitly, the decompositions (in the complex field) (5.6) and (5.7) still hold, but the time-like reduction interpretation is different, because it here concerns the tripartite entanglement of seven qubits which are split, on the real field, into four qubits transforming under $\mathrm{SU}(2)$ and three qubits transforming under $\mathrm{SL}(2, \mathbb{R})$ qubits, the fourth $\mathrm{SL}(2, \mathbb{R})$ being the Ehlers group (the very same commuting with $E_{7(-25)}$ inside $E_{8(-24)}$, see (6.3)). In this case, it holds that

$$\mathfrak{e}_{8(-24)} \cong ([\mathfrak{sl}(2, \mathbb{R})]^4 + [\mathfrak{su}(2)]^4) \oplus \mathfrak{p}, \quad (6.4)$$

where \mathfrak{p} has the same formal decomposition as given in (5.9), but with the second quartet of irreps. pertaining to $[\mathrm{SU}(2)]^4$, and not to $[\mathrm{SL}(2, \mathbb{R})]^4$ as in the maximal case. This admits an interpretation as the four-way entanglement of eight qubits, democratically covariant with respect to the two possible symmetry groups of SLOCC-equivalent real qubits, namely four with respect to $\mathrm{SU}(2)$ and four with respect to $\mathrm{SL}(2, \mathbb{R})$. This is a rather weird split combination from the QIT point of view, and we leave for future investigation the question of whether this setup enjoys any real use.

While one can formally still assign $|\Psi\rangle_{224}$ to the coset

$$\frac{E_{8(-24)}}{[\mathrm{SL}(2, \mathbb{R})]^4 \times [\mathrm{SU}(2)]^4}, \quad (6.5)$$

unlike the STU example and analogously to the $\mathcal{N} = 8$ case treated in previous Section, the Kostant-Sekiguchi theorem does *not* apply, because (6.5) is not a symmetric space (as it can be verified e.g. by considering the non-zero commutation relations

¹³Note that $E_{8(-24)}$ contains (in a maximal and symmetric way, see e.g. [55]) $\mathrm{SO}^*(16)$, but the c^* -map determines the relevant subgroup to be $E_{7(-25)} \times \mathrm{SL}(2, \mathbb{R})$. From a physical perspective, this can ultimately be related to the split between the gravity and vector multiplets, which is not present in the maximal theory treated in previous Section.

of, for example, two elements in $(2, 2, 2, 1, 2, 1, 1, 1)$ and $(2, 1, 2, 2, 1, 2, 1, 1)$). Once again, this can be traced back to the mismatching between the $[\text{SL}(2)]^7$ group and the whole $D = 4$ U-duality group (in the complex field); indeed, in $D = 4$ the tripartite entanglement of seven qubits forms a representation of the full U-duality group, not just its $[\text{SL}(2)]^7$ subgroup. Consequently, performing the time-like reduction, it is actually the nilpotent orbits of $E_7(\mathbb{C}) \times \text{SL}(2, \mathbb{C})$ acting on the irrepr. $(56, 2)$ that are of relevance.

Acknowledgements

We would like to thank Sergio Ferrara and Philip Gibbs for useful conversations. The work of LB is supported by ERC advanced grant no. 226455, *Supersymmetry, Quantum Gravity and Gauge Fields* (SUPERFIELDS). The work of MJD is supported by the STFC under rolling grant ST/G000743/1.

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